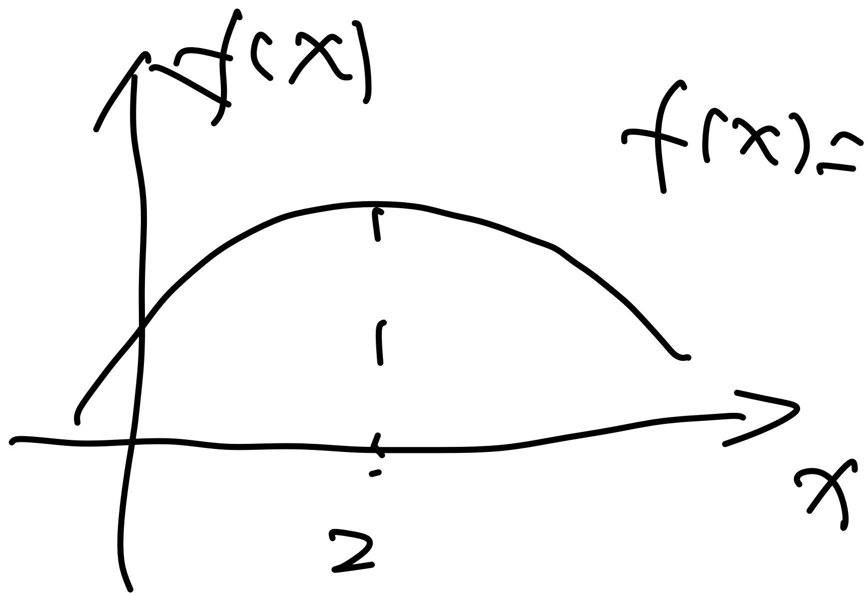
Lecture 2

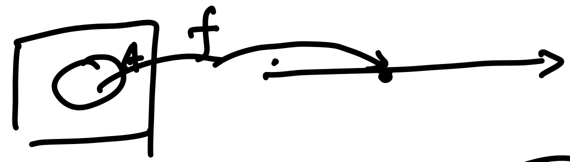
Longhai Li, September 9th, 2021

Set function

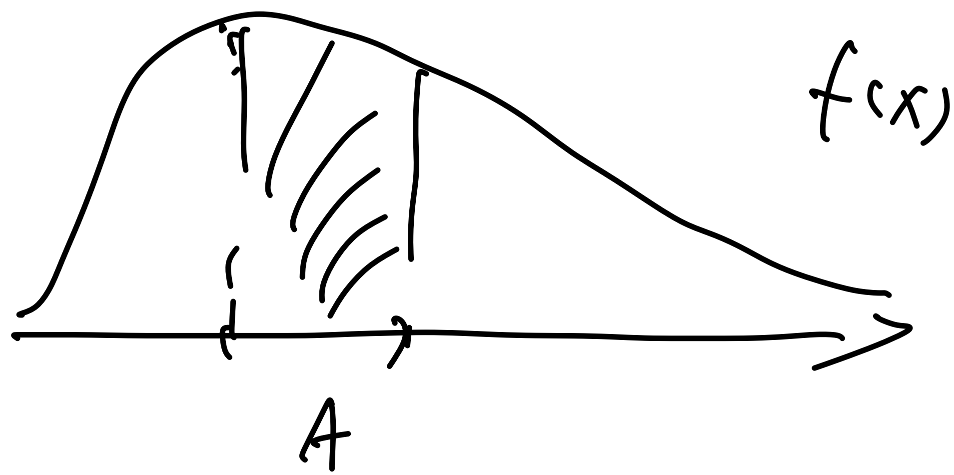
A function that assigns a number to a set.



$$f(x) = -(x-2)^2$$



e.g.

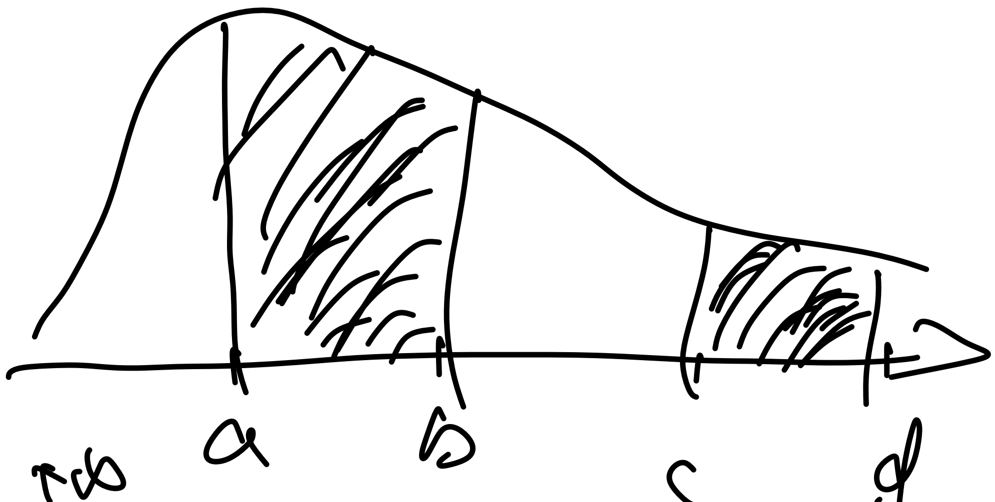


$Q(A) =$ area under the curve of $f(x)$.

$A = [a,b]$. $\quad Q(A) = \int_a^b f(x)\,dx$

$A = [a,b] \cup [c,d]$

$$Q(A) = \int_a^b f(x)\,dx + \int_c^d f(x)\,dx$$

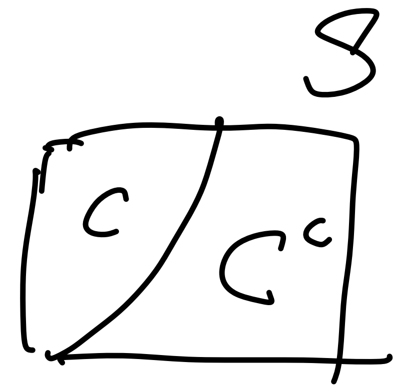


$$A = \bigcup_{i=1}^{+\infty} A_i \quad , \quad Q(A) = \sum_{i=1}^{+\infty} \int_{A_i} f(x)\,dx$$

Thm 1.3.2.

$$P(C) + P(C^c) = 1$$

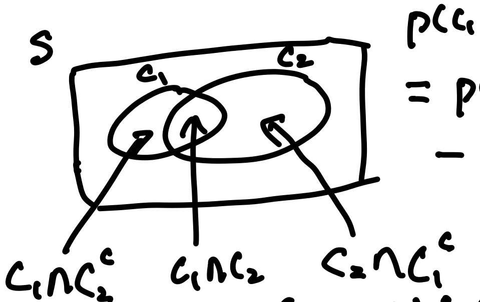
Pf: $S = C \cup C^c$



$$\begin{cases} P(S) = 1 \\ P(C \cup C^c) = P(C) + P(C^c) \end{cases}$$

$$\therefore \quad 1 = P(C) + P(C^c)$$

Thm 1.3.5



$P(C_1 \cup C_2)$
$= P(C_1) + P(C_2)$
$- P(C_1 \cap C_2)$

$A \cap B = AB$

$C_1 \cup C_2 = (C_1 \cap C_2^c) \dot{\cup} (C_1 \cap C_2) \dot{\cup} (C_2 \cap C_1^c)$

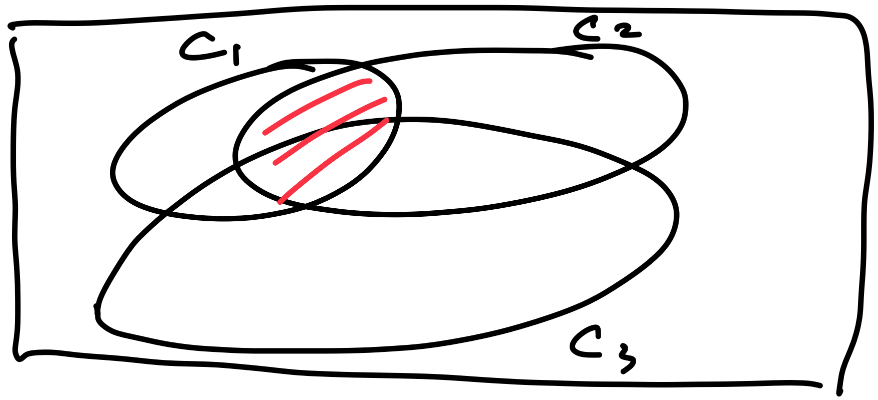
$$P(C_1 \cup C_2) = P(C_1 C_2^c) + P(C_1 C_2) + P(C_1^c C_2)$$

$$= P(C_1 C_2^c) + P(C_1 C_2) \rightarrow P(C_1)$$

$$P(C_1^c C_2) + P(C_1 C_2) \rightarrow P(C_2)$$

$$- P(C_1 C_2)$$

$$= P(C_1) + P(C_2) - P(C_1 C_2)$$

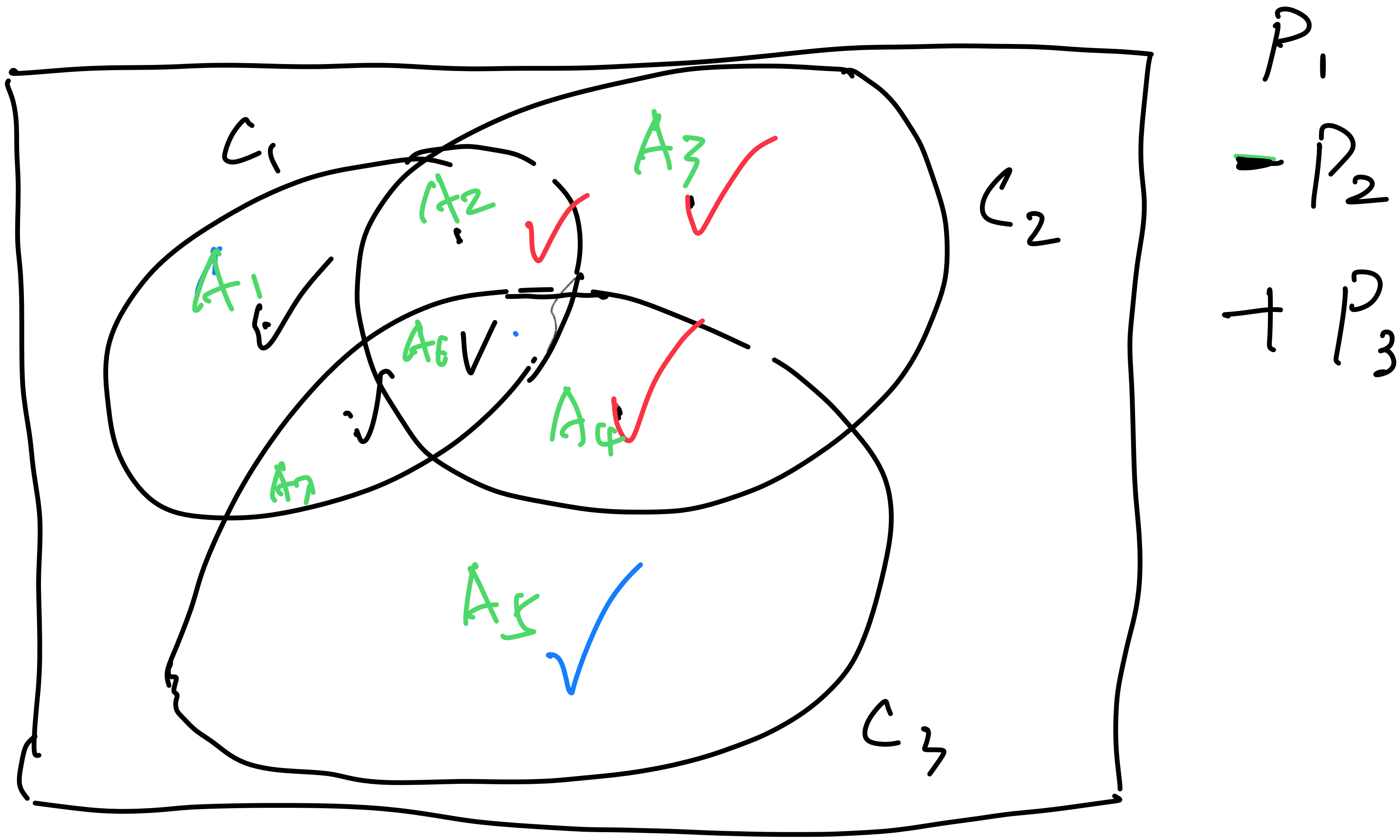


$$P_1 = P(C_1) + P(C_2) + P(C_2)$$

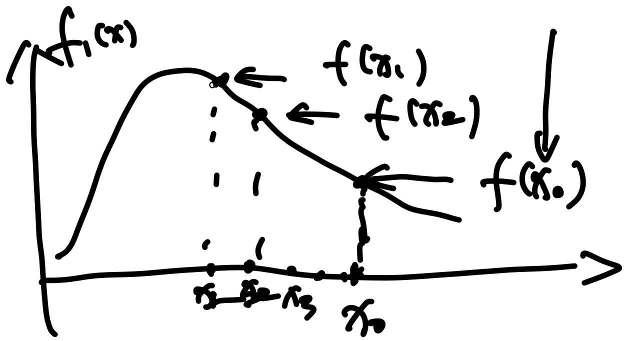
$$P_2 = P(C_1C_2) + P(C_2C_3) + P(C_1C_3)$$

$$P_3 = P(C_1C_2C_3)$$

$$P(C_1 \cup C_2 \cup C_3) = P_1 - P_2 + P_3$$


C1
C2
C3
A1
A2
A3
A4
A5
A6
A7
P1
- P2
+ P3

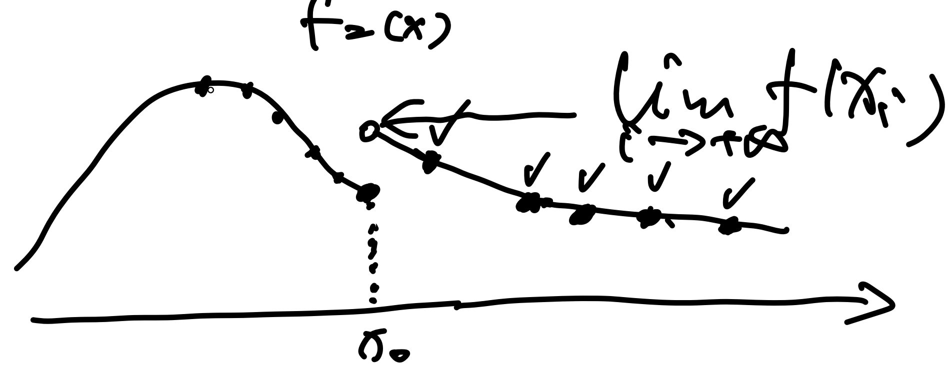
Continuity of a function



If $x_i \searrow x_0$ $\lim_{i\to+\infty} x_i = x_0$

$$\lim_{i\to+\infty} f(x_i) = f(x_0) = f(\lim_{i\to+\infty} x_i)$$

f is cont. at x_0



If $x_i \nearrow x_0$,

$$\lim_{i\to+\infty} f(x_i) = f(x_0) \quad \checkmark$$

If $x_i \searrow x_0$

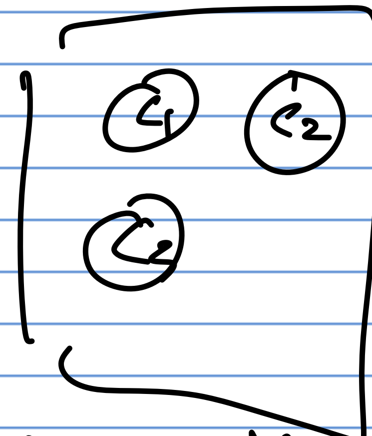
$$\lim_{i\to+\infty} f(x_i) \neq f(x_0)$$

Axioms of prob:

Continuity of Prob

$C_1, C_2, \ldots$ mutually exclusive

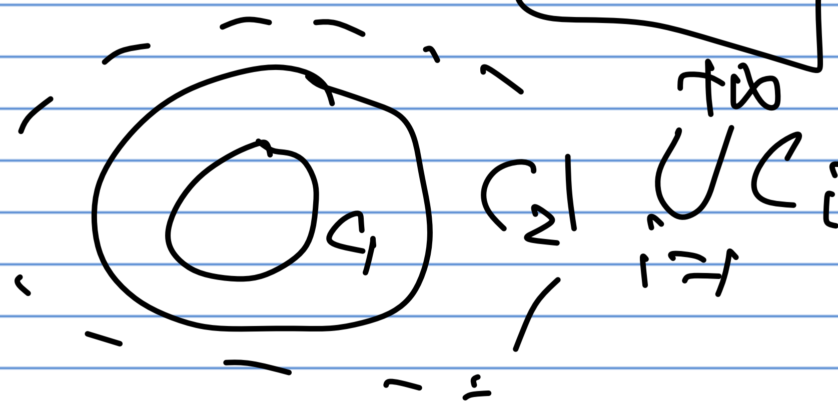
$$P\left(\bigcup_{i=1}^{+\infty} C_i\right) = \sum_{i=1}^{+\infty} P(C_i)$$



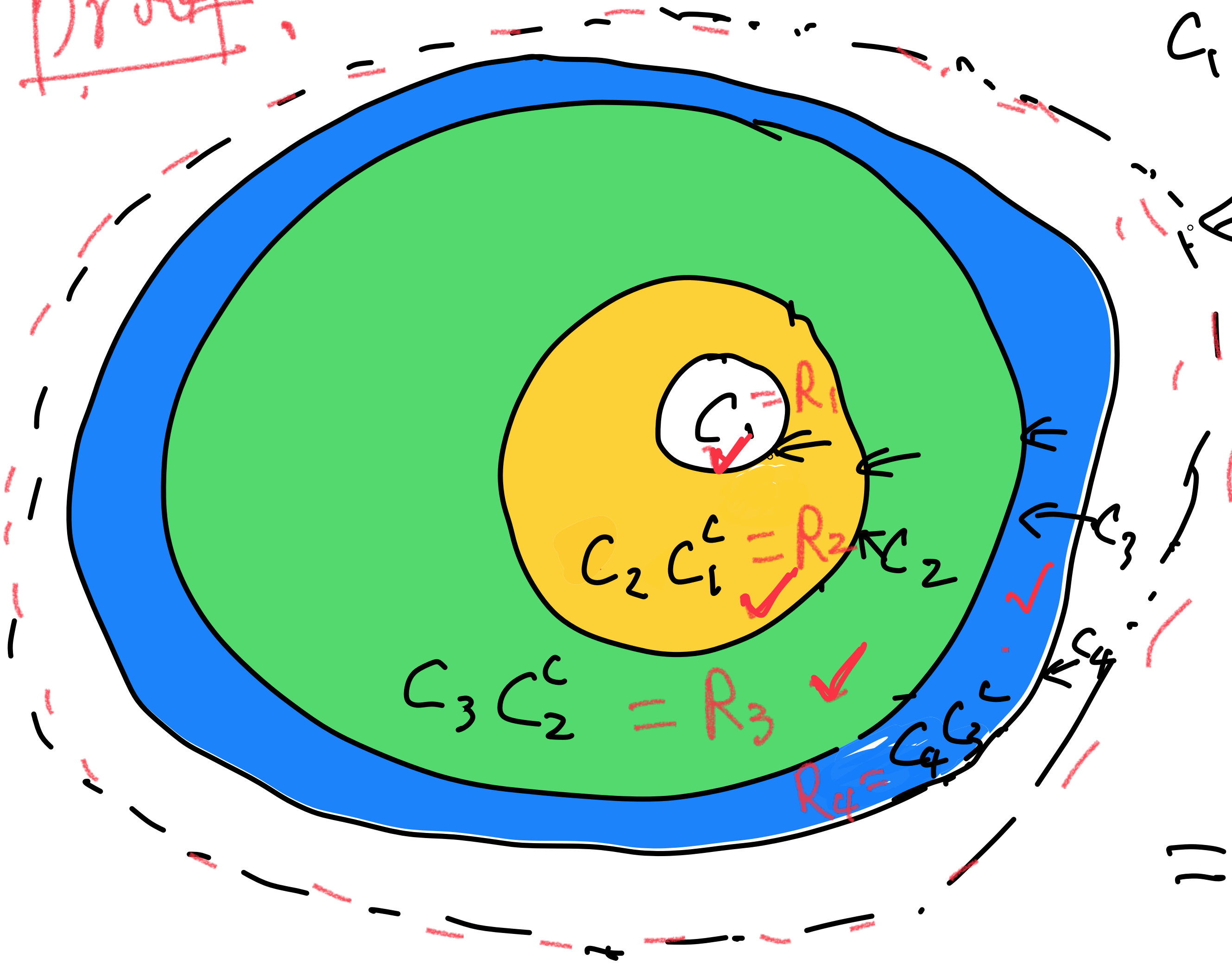
$\Leftrightarrow$

$$C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots$$

$$\lim_{i \to +\infty} P(C_i) = P\left(\lim_{i \to +\infty} C_i\right)$$



proof:



$C_1 \subseteq C_2 \subseteq \cdots\cdots$

$\lim_{i \to +\infty} C_i$

$= \bigcup_{i=1}^{+\infty} C_i$

$P(\lim_{i \to +\infty} C_i)$

$= \lim_{i \to +\infty} P(C_i)$

$$\lim_{i\to+\infty} C_i = \bigcup_{i=1}^{+\infty} C_i = \underbrace{C_1}_{R_1} \,\dot\cup\, \underbrace{C_2 C_1^c}_{R_2} \,\dot\cup\, \underbrace{C_3 C_2^c}_{R_3} \,\dot\cup\, \cdots$$

$$P\Big(\bigcup_{i=1}^{+\infty} C_i\Big) = \sum_{i=1}^{+\infty} P(\underbrace{C_i C_{i-1}^c}_{R_i}), \quad C_0 = \phi$$

$$P\big(\lim_{i\to+\infty} C_i\big) = \sum_{i=1}^{+\infty} P(\underbrace{C_i C_{i-1}^c}_{R_i})$$

$$\stackrel{?}{=} \lim_{i\to+\infty} P(C_i)$$

$$\sum_{i=1}^{+\infty} P(C_i C_{i-1}^c)$$

$$= \sum_{i=1}^{+\infty} \left[P(C_i) - P(C_{i-1}) \right]$$

$$= \lim_{n \to +\infty} \sum_{i=1}^{n} \left[P(C_i) - P(C_{i-1}) \right]$$

$$= \lim_{n \to +\infty} \left[\begin{array}{l} \cancel{P(C_1)} - P(C_0) + \\ \cancel{P(C_2)} - \cancel{P(C_1)} + \\ \quad \vdots \\ P(C_n) - \cancel{P(C_{n-1})} \end{array} \right]$$

$$= \lim_{n \to +\infty} P(C_n), \quad \text{Note } P(C_0) = 0$$

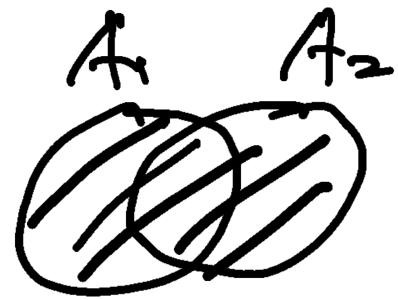
$P(C_i C_{i-1}^c) = P(C_i) - P(C_{i-1})$

$$C_{i-1} \cup R_i = C_i$$

Boole's inequality

$$P\left(\bigcup_{i=1}^{+\infty} A_i\right) \le \sum_{i=1}^{+\infty} P(A_i)$$

called sub-additivity.

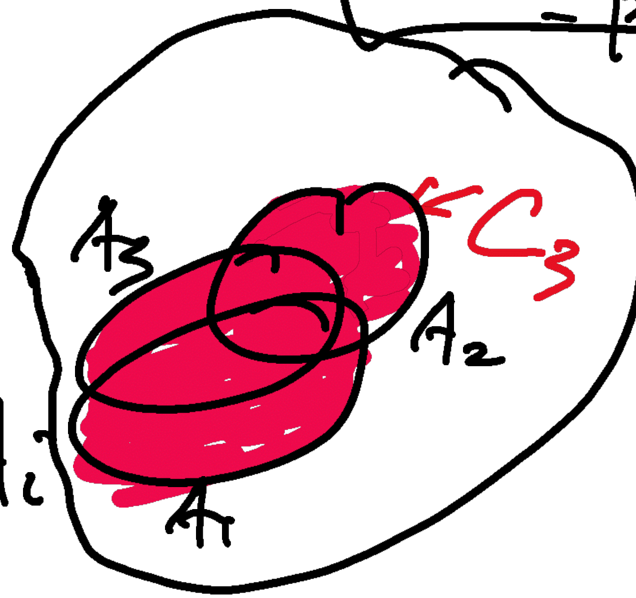


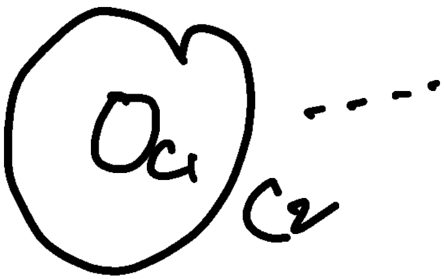
$P(A_1 \cup A_2) \le P(A_1) + P(A_2)$

Pf:

$$C_i = \bigcup_{k=1}^{i} A_k$$

$$C_i \uparrow \quad \bigcup_{i=1}^{+\infty} C_i = \bigcup_{i=1}^{+\infty} A_i$$





$$P(A_1 \cup \cdots \cup A_k) \le P(A_1) + \cdots + P(A_k)$$

$$P(C_i) = P\left(\bigcup_{k=1}^{i} A_k\right) \le \sum_{k=1}^{i} P(A_k)$$

So, $$P\left(\bigcup_{i=1}^{+\infty} A_i\right) = \lim_{i \to +\infty} P(C_i) \le \lim_{i \to +\infty} \sum_{k=1}^{i} P(A_k) = \sum_{k=1}^{+\infty} P(A_k)$$

End of proof.