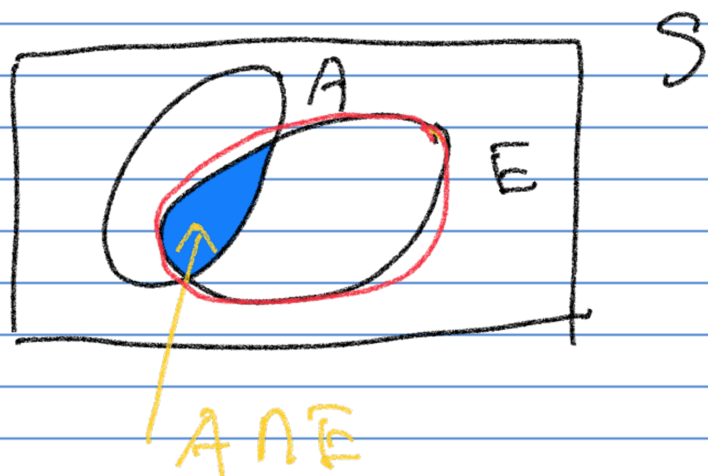
Lecture 3

Longhai Li, September 14, 2021

Conditional prob & Indep. (Sec 1.4)

Def: Assume $P(E) > 0$,

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)}.$$



Example:

H: going to hospital due to covid

V: receiving a vaccine.

	H	H^c	
V	100	700	800
V^c	150	50	200
	250	750	

$$P(H) = \frac{250}{1000} = \frac{1}{4}$$

$$P(H \mid V) = \frac{P(H \cap V)}{P(V)} = \frac{100}{800} = \frac{1}{8} \quad \checkmark$$

$$P(H \mid V^c) = \frac{150}{200} = \frac{3}{4} \quad \checkmark$$

$$P(V \mid H) = \frac{P(V \cap H)}{P(H)} = \frac{100}{250} = 0.4$$

$$P(V^c \mid H) = \frac{150}{250} = 0.6$$

$P(V \mid H) = 0.4$ $\qquad$ $P(V^c \mid H) = 0.6$

Condit. prob satisfies prob. axioms:

$P(E) > 0$,

1) $P(C_i \mid E) \geq 0$

2) $P(S \mid E) = 1$

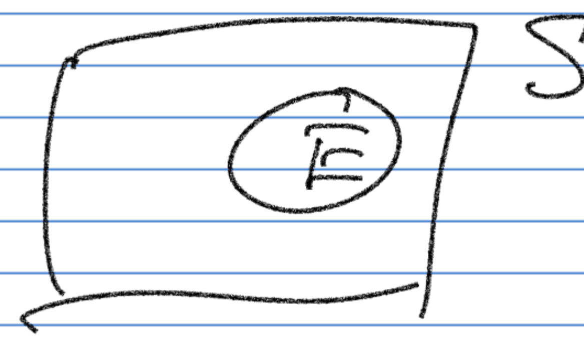
3) If $C_1, C_2, \ldots$ are mutually exclusive

$$P\left(\bigcup_{i=1}^{+\infty} C_i \,\Big|\, E\right) = \sum_{i=1}^{+\infty} P(C_i \mid E)$$

proof:

1) $$P(C_i \mid \bar{E}) = \frac{P(C_i \bar{E})}{P(\bar{E})} \geq 0$$

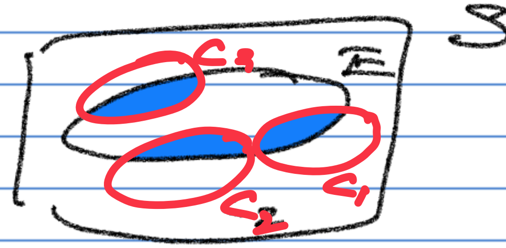
2) $$P(S \mid \bar{E}) = \frac{P(S \cap \bar{E})}{P(\bar{E})} = \frac{P(\bar{E})}{P(\bar{E})} = 1$$



3) $C_1, C_2, \cdots$ are mutually exclusive.

$$P\left(\bigcup_{i=1}^{+\infty} C_i \mid E\right)$$

$$= \frac{P\left(\left(\bigcup_{i=1}^{+\infty} C_i\right) E\right)}{P(E)}$$



$$= \frac{P\left(\bigcup_{i=1}^{+\infty} (C_i E)\right)}{P(E)}$$

$$= \frac{\sum_{i=1}^{+\infty} P(C_i E)}{P(E)}$$

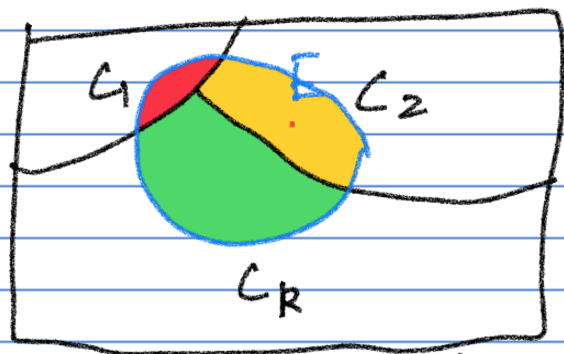
$C_1E, C_2E, C_3E, \cdots$ are mutually exclusive.

$$= \sum_{i=1}^{+\infty} \frac{P(C_i E)}{P(E)} = \sum_{i=1}^{+\infty} P(C_i \mid E)$$

Multiplication Rule

$$P(C_1 \cap C_2) = P(C_1) \cdot P(C_2 \mid C_1)$$

Law of Total prob.



$\mathcal{S}$ $\quad \bigcup_{i=1}^{k} C_i = \mathcal{S}$

$C_1, \ldots, C_k$ M. E.

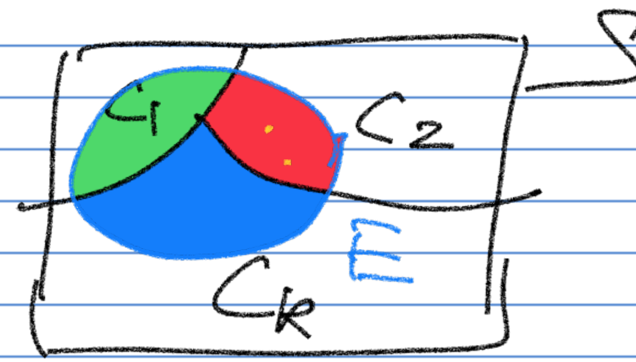
$$P(E) = \sum_{i=1}^{k} P(C_i) P(E \mid C_i)$$

Pf: $\quad E = \bigcup_{i=1}^{k} (C_i E)$

$$P(E) = \sum_{i=1}^{k} P(C_i E) = \sum_{i=1}^{k} P(C_i) \cdot P(E \mid C_i)$$

Bayes Rule

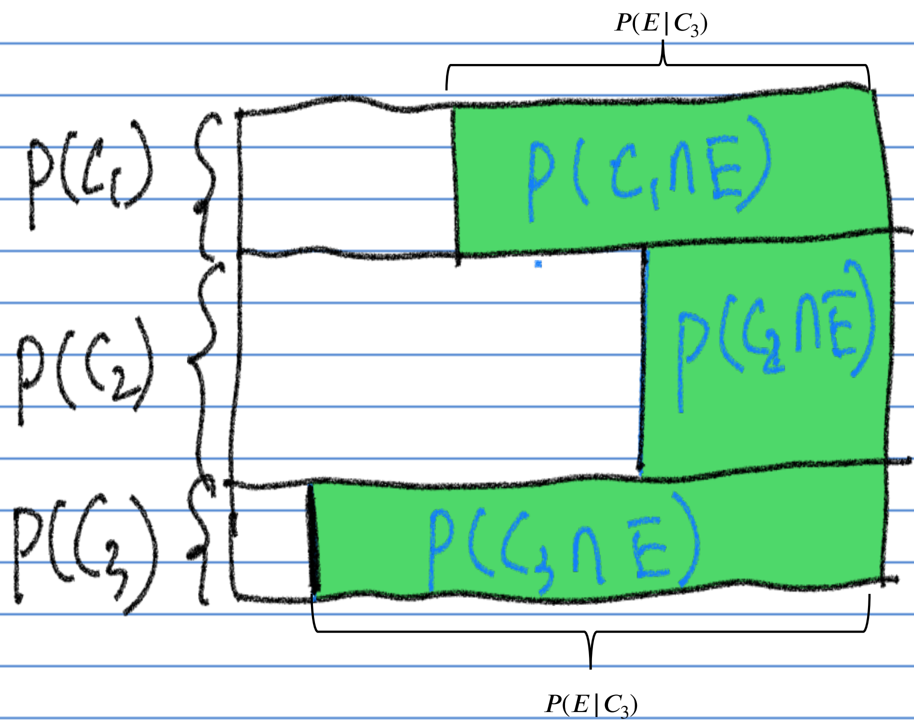
$$\bigcup_{i=1}^{k} C_i = S, \quad C_i \cap C_j = \emptyset$$



$$p(C_i \mid E) = \frac{p(C_i \cap E)}{p(E)} = \frac{p(C_i) \cdot p(E \mid C_i)}{\sum_{i=1}^{k} p(C_i)\, p(E \mid C_i)}$$

$$p(C_i \mid E) \propto p(C_i)\, p(E \mid C_i)$$

A graph to illustrate Bayes rule



$$P(C_i | E) :$$

$$\propto P(C_i) \cdot P(E | C_i)$$

prior — causal

Also, $P(E | C_i) \propto \dfrac{P(C_i | E)}{P(C_i)}$

Two extreme cases:

1) $p(C_1) = p(C_2) = \cdots = p(C_k)$

$$p(C_i \mid E) \propto p(E \mid C_i)$$

2) $p(E \mid C_1) = p(E \mid C_2) = \cdots = p(E \mid C_k)$

$$p(C_i \mid E) \propto p(C_i)$$

Example:

	H	H^c	
V_0	150	50	200
V_1	100	700	800
	250	750	

V_0: No vaccined

V_1: Vaccined.

H: going to hospital due to covid 19

	V_0	V_1
$P(V_i)$	0.2	0.8
$P(H\mid V_i)$	$\frac{3}{4}$	$\frac{1}{8}$
$P(V_i)\cdot P(H\mid V_i)$	0.2×0.75	0.8×0.125

$\longrightarrow$ $P(H) = 0.2\times 0.75 + 0.8\times 0.125$
$= 0.25$

$$P(V_i\mid H) \propto P(V_i)\cdot P(H\mid V_i)$$

$$P(V_0\mid H) = \frac{0.2\times 0.75}{0.25} = 0.2\times 3 = 0.6$$

$$P(V_1\mid H) = \frac{0.8\times 0.125}{0.25} = 0.4$$

Another way to find $p(V_i \mid H)$:

$$\frac{p(V_1 \mid H)}{p(V_0 \mid H)} = \frac{p(V_1) \cdot p(H \mid V_1)}{p(V_0)\, p(H \mid V_0)}$$

$$= \frac{0.8 \times \frac{1}{8}}{0.2 \times \frac{3}{4}}$$

$$= \frac{4 \times 1}{1 \times 6} = \frac{2}{3}$$

$p(V_1 \mid H) = \frac{2}{5} = 0.4 \qquad p(V_0 \mid H) = \frac{3}{5} = 0.6$

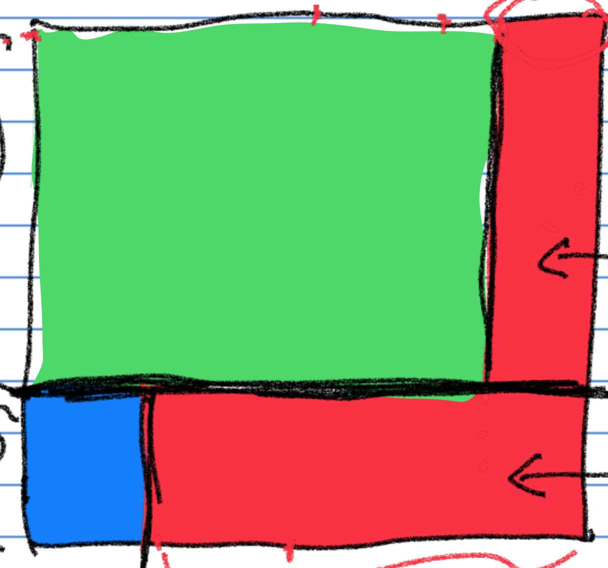
A graph to illustrate the previous calculation

$P(H \mid V_1)$

H: going to hospital due to covid

$P(V_1) = 0.8$

$P(V_0) = 0.2$



$P(H \cap V_1) = 0.8 \times \frac{1}{8}$

$P(H \cap V_0) = 0.2 \times \frac{2}{4}$

$P(H \mid V_0)$

$$P(V_i \mid H) \propto P(V_i) \cdot P(H \mid V_i)$$

$$P(H \mid V_i) \propto \frac{P(V_i \mid H)}{P(V_i)}$$

Finding the ratio of causal Probabilities given prior and posterior probabilities

Suppose we see that $\frac{P(V_1|H)}{P(V_0|H)} = \frac{2}{1}$ & $\frac{P(V_0)}{P(V_1)} = \frac{2}{8}$

$\frac{P(H|V_1)}{P(H|V_0)} = ?$ using Bayes rule.

$$\frac{P(V_1|H)}{P(V_0|H)} = \frac{P(V_1)}{P(V_0)} \cdot \frac{P(H|V_1)}{P(H|V_0)}$$

$$\frac{P(H|V_1)}{P(H|V_0)} = \frac{P(V_1|H)}{P(V_0|H)} \cdot \frac{P(V_0)}{P(V_1)}$$

$$\frac{P(H|V_1)}{P(H|V_0)} = \frac{2}{1} \times \frac{2}{8} = \frac{4}{8} = \frac{1}{2}$$

We can see that when $\frac{P(V_1|H)}{P(V_0|H)} > \frac{P(V_1)}{P(V_0)}$

$\frac{P(H|V_1)}{P(H|V_0)}$ is > 1, o.w. < 1 (i.e., vaccination is useful) public data

A Real Numerical Example

dashboard.saskatchewan.ca/health-wellness/covid-19/cases

Highlights

- As of September 20th, there are 519 new confirmed cases of COVID-19, bringing the total to 62,620 reported cases.
- The new cases are located in the Far North West (23), Far North Central (1), Far North East (21), North West (63), North Central (53), North East (20), Saskatoon (136), Central West (10), Central East (33), Regina (54), South West (21), South Central (26) and South East (19) zones and thirty nine (39) new cases have pending residence information.
- Thirteen (13) cases with pending residence information were reassigned to Far North West (from September 8 (1), September 16 (1)), Far North East (from September 11 (1), September 17 (1)), North Central (from September 17 (2)), North East (from September 17 (1)), Saskatoon (from September 4 (1), September 5 (1), September 18 (2)), South Central (from September 18 (1)), and South East (from September 16 (1)) zones.
- 62,620 cases are confirmed
  - 15,146 cases are from the Saskatoon area
  - 15,019 cases are from the North area (6,307 North West, 6,381 North Central, 2,331 North East)
  - 13,054 cases are from the Regina area
  - 7,747 cases are from the Far North area (3,623 Far North West, 522 Far North Central, 3,602 Far North East)
  - 7,163 cases are from the South area (1,488 South West, 2,266 South Central, 3,409 South East)
  - 4,034 cases are from the Central area (1114 Central West, 2,920 Central East)
  - 457 cases have pending residence information
- 4,672 cases are considered active and 57,307 cases are considered recovered.
- Nearly one-third (32.9%) of new cases are in the age category of 20-39.
- Approximately one in five (19.3%) of new cases eligible for vaccination (aged 12 years and older) were fully vaccinated.
- As of September 20th, a total of 253 individuals are hospitalized; including 197 inpatient hospitalizations and 56 ICU hospitalizations. Of the 253 patients, 189 (74.7%) were not fully vaccinated.
- Two (2) new deaths reported today. 641 Saskatchewan residents with COVID-19 have died with a case fatality rate of 1.0%
- 1,094,674 COVID-19 tests have been performed in the province. As of September 16th, 2021, when other provincial and national numbers are available from PHAC, Saskatchewan's per capita rate was 914,105 tests performed per million. The national rate was 1,104,415 tests performed per million.
- The 7-day average of new COVID-19 case numbers was 494 (41.0 new cases per 100,000).

Released: September 20, 2021

C : **infected by covid19**

$$P(V_0) = 1/4,\ P(V_1) = 3/4$$

$$P(V_0 \mid C) = 4/5,\ P(V_1 \mid C) = 1/5$$

$$\frac{P(C \mid V_1)}{P(C \mid V_0)} = \frac{1/3}{4/1} = 1/12$$

$$\frac{1/5}{3/4} \quad \frac{4/1}{1/4}$$

vac's efficacy in preventing infection $= 11/12 = 92\,\%$

H : **hospitalized due to covid19**

$$P(V_0 \mid H) = 3/4,\ P(V_1 \mid H) = 1/4$$

$$\frac{P(H \mid V_1)}{P(H \mid V_0)} = \frac{1/3}{3/1} = 1/9$$

vac's efficacy in preventing hospitalization $= 8/9 = 89\,\%$