Lecture 4
Independence of Events

Longhai Li, September 16, 2021

Independence

Definition :

C_1 & C_2 are indep if

- $p(C_1 \cap C_2) = p(C_1) \times p(C_2)$ ✓
- $p(C_1 \mid C_2) = \dfrac{p(C_1 \cap C_2)}{p(C_2)} = \dfrac{p(C_1) \times \cancel{p(C_2)}}{\cancel{p(C_2)}}$

$$= p(C_1)$$

C_2 doesn't alter the chance of C_1

Example

	H	H^c
V_0	0.8	0.2
V_1	0.8	0.2

$\Leftarrow p(H \mid V_i)$

$$p(H \mid V_0) = p(H \mid V_1) = p(H)$$

$$p(H) = p(V_0) \times p(H \mid V_0) + p(V_1)\, p(H \mid V_1)$$

$$= p(V_0) \times 0.8 + p(V_1) \cdot 0.8 = 0.8$$

H & V_0 indep, H & V indep

Example

	H	H^c
V_0	0.8	0.2
V_1	0.1	0.9

$\leftarrow P(H \mid V_i)$

$P(V_0) = 0.8, \ P(V_1) = 0.2$

$$P(H \mid V_0) \neq P(H \mid V_1)$$

$$P(H) = P(V_0) \cdot P(H \mid V_0) + P(V_1) \cdot P(H \mid V_1)$$

$$= 0.2 \times 0.8 + 0.8 \times 0.1$$

$$= 0.16 + 0.08 = 0.24$$

$$P(H \mid V_0) = 0.8 \neq P(H)$$

H & V_0 are dependent.

Remark 1:

ϕ is indep of any set A

$p(\phi \mid A) = 0$, $p(\phi) = 0$

$p(\phi \cap A) = p(\phi) = 0$

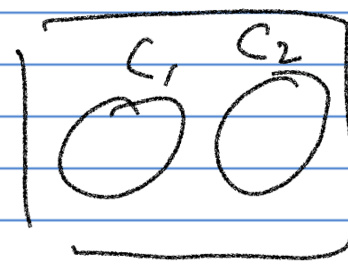
$p(\phi) = 0$

S is indep of any set A

Remark 2:

Indep $\neq$ mutually exclusive.

$C_1 \cap C_2 = \emptyset$



$P(C_1 \cap C_2) = 0$

$\neq P(C_1) \cdot P(C_2)$ if $P(C_1) > 0$, $P(C_2) > 0$

$$P(C_1 \mid C_2) = \frac{P(C_1 \cap C_2)}{P(C_2)} = \frac{0}{P(C_2)} = 0 \neq P(C_1)$$

M.E. events are strongly dependent.

Indep of $C_1, \cdots, C_n$:

Given any subset $i_1, i_2, \cdots, i_k \in \{1, \cdots, n\}$

for any $k \in \{1, \cdots, n\}$,

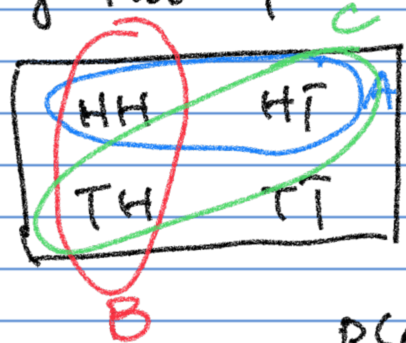
$$P(C_{i_1} \cap C_{i_2} \cdots \cap C_{i_k})$$

$$= P(C_{i_1}) \cdot P(C_{i_2}) \cdots P(C_{i_k})$$

Remark: pair-wise indep is not suff.

Example :

Flipping two fair coins : $p(\omega) = \frac{1}{4}$



A: 1st coin is H.

B: 2nd coin is H

C: two coins show diff. outcomes.

$$P(A) = P(B) = P(C) = \frac{1}{2}$$

$$P(AB) = P(BC) = P(AC) = \frac{1}{4}$$

$$P(AB) = P(A) \cdot P(B)$$

$$P(BC) = P(B) \cdot P(C)$$

$$P(AC) = P(A) \cdot P(C)$$

A, B, C are pairw. indep.

$$A \cap B \cap C = \emptyset \qquad A \cap B \Rightarrow C^c$$

$$P(ABC) \neq P(A) \cdot P(B) \cdot P(C)$$

Example:

Flipping three coins independently.

$$S = \left\{ \begin{array}{l} (1\ 1\ 1), \\ (1\ 1\ 0), \\ (1\ 0\ 1), \\ (1\ 0\ 0), \\ (0\ 1\ 1), \\ (0\ 1\ 0), \\ (0\ 0\ 1), \\ (0\ 0\ 0) \end{array} \right\} \qquad P(\{w\}) = \frac{1}{8}, \text{ for } w \in S$$

H_1: 1st coin shows H

H_2: 2nd coin shows H

H_3: 3rd coin shows H

$$P(H_i) = \frac{1}{2}$$

$$P(H_1 \cap H_2) = P\left(\left\{ \begin{array}{l} (1,1,0), \\ (1,1,1) \end{array} \right\}\right)$$

$$= \frac{1}{4}$$

$$P(H_i \cap H_j) = \frac{1}{4} = P(H_i) \cdot P(H_j)$$

$$H_1 \cap H_2 \cap H_3 = \{(1,1,1)\}$$

$$P(H_1 H_2 H_3) = \frac{1}{8} = P(H_1) \cdot P(H_2) P(H_3)$$

H_1, H_2, H_3 are indep.