Lecture 5

Longhai Li, September 21, 2021

$C_1, C_2, \cdots, C_n$ indep.:

$$\bar{P}(C_{i_1} \cap C_{i_2} \cdots \cap C_{i_k}) = \prod_{j=1}^{k} P(C_{i_j})$$

A and B are indep

$\Leftrightarrow$ A & B^c indep

$\Leftrightarrow$ A^c & B indep

$\Leftrightarrow$ A^c & B^c indep.

Pf:

$P(A\cap B) = P(A)\,P(B)$

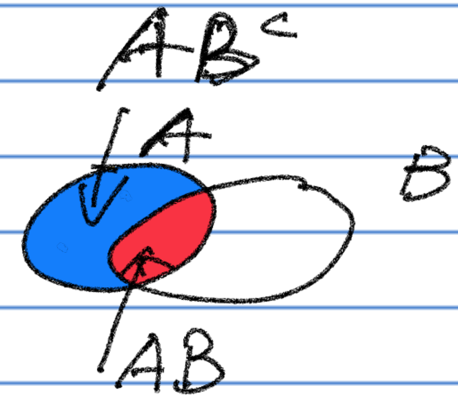
$P(A\cap B^c)$

$= P(A) - P(AB)$

$= P(A) - P(A)\cdot P(B)$

$= P(A)\cdot(1 - P(B))$

$= P(A)\,P(B^c)$



$A = AB \,\dot{\cup}\, AB^c$

$P(A) = P(AB) + P(AB^c)$

$A \perp B \Rightarrow A \perp B^c \Rightarrow A^c \perp B^c$

C_1, C_2, C_3, C_4 are indep $\Longleftrightarrow$

C_1^c, C_2^c, C_3, C_4 all indep?

Example: (multiple comparison bias)

$R_1, \dots, R_n$ indep., $n > 1$

$P(R_i) = 0.05$, $P(R_i^c) = 0.95$

$P(\text{at least one } R_i \text{ occur})$

$$= P(R_1 \cup R_2 \cup R_3 \cup \dots \cup R_n)$$

$$= 1 - P\left(\left(\bigcup_{i=1}^{n} R_i\right)^c\right)$$

$$= 1 - P\left(\bigcap_{i=1}^{n} R_i^c\right)$$

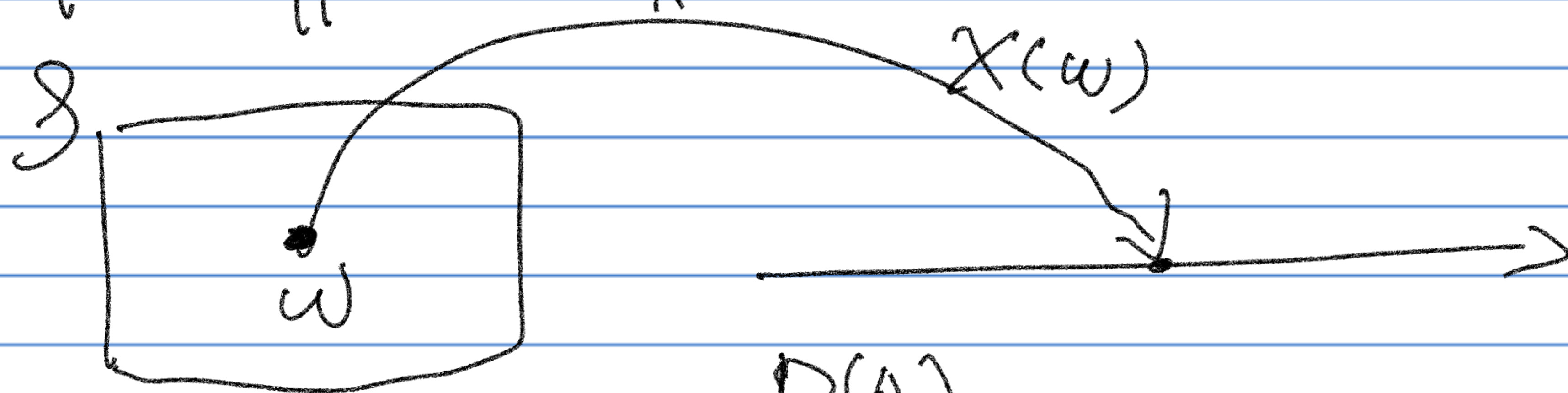
$$= 1 - \prod_{i=1}^{n} P(R_i^c) \quad [\text{indep assumption}]$$

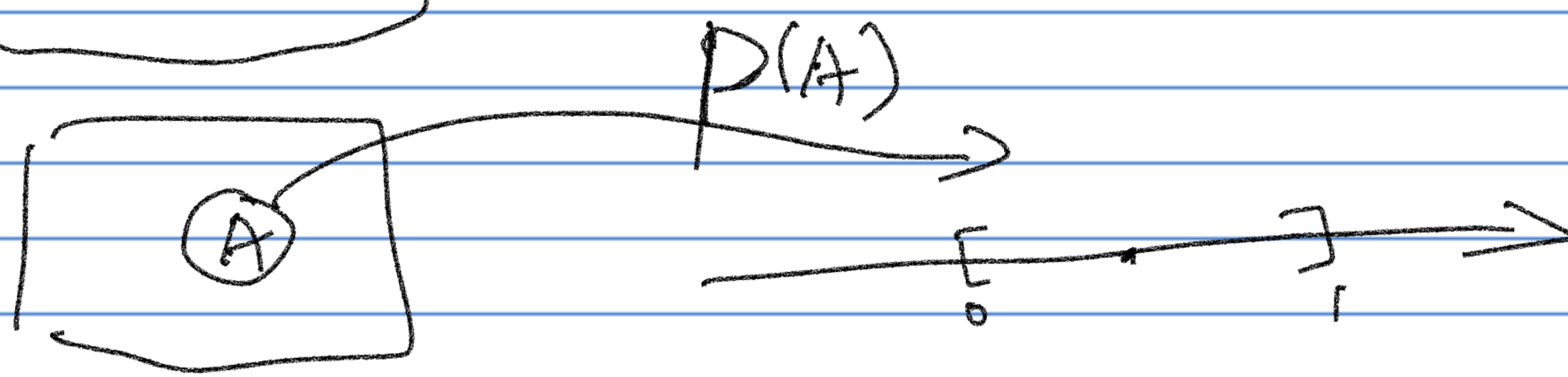
$$= 1 - 0.95^n = \underline{1 - (1 - 0.05)^n} \approx \underline{0.05 \times n}$$

$n = 10$, $1 - 0.95^{10} = 0.4$

Random Variables & C.D.F.

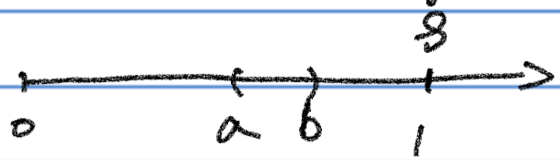
Def: A r.v. X is a function from sample space to a real value

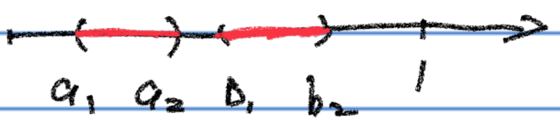




Examples :

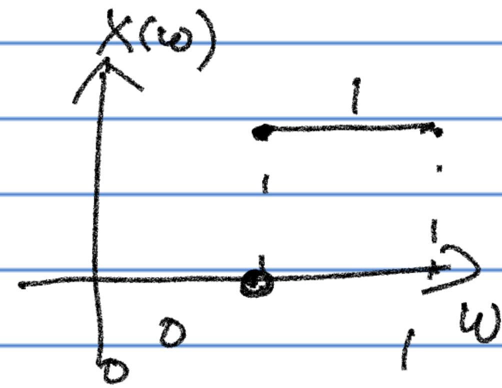
$S = [0,1]$. $P((a,b)) = b-a$, $P(\{\omega\}) = 0$





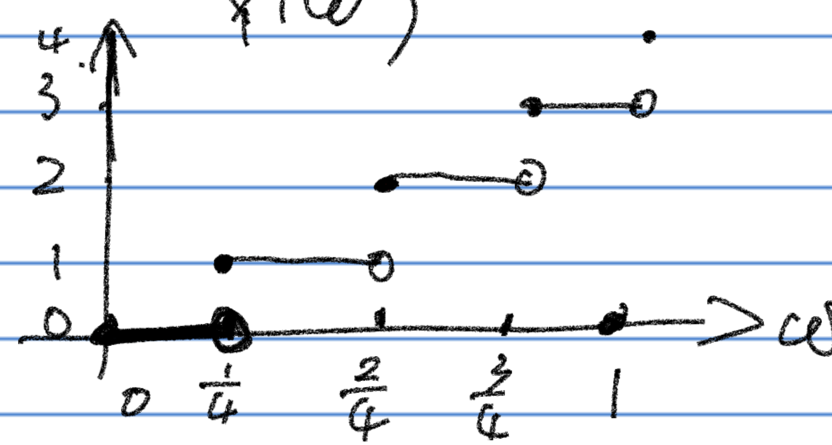
$$X(\omega) = \begin{cases} 0, & \text{if } \omega < \frac{1}{2} \\ 1, & \text{if } \omega \geq \frac{1}{2} \end{cases}$$

$$= \mathbb{1}(\omega \geq \tfrac{1}{2})$$



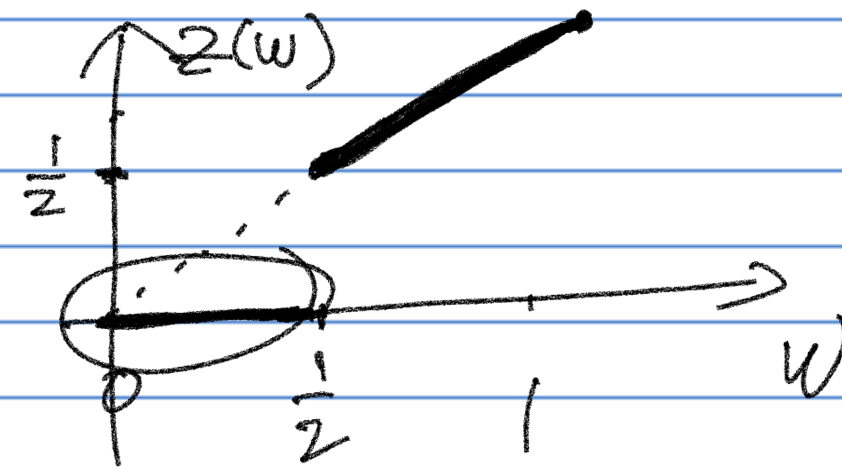
binary

$Y(\omega) = \lfloor 4\omega \rfloor$ = floor of 4ω



X and Y are discrete

$$Z(w) = \begin{cases} 0, & w < \frac{1}{2} \\ w, & w \geq \frac{1}{2} \end{cases}$$



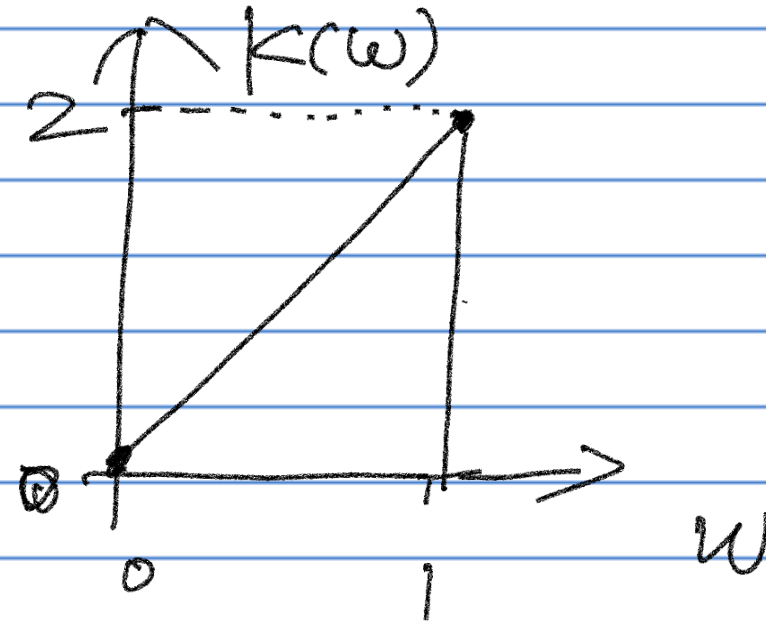
Z is a mixed r.v.

$$P(Z = 0) = P(0 < w < \tfrac{1}{2}) = \tfrac{1}{2}$$

$$P(Z = 0.75) = P(w = 0.75) = 0$$

Range of $Z = \{0\} \cup [\tfrac{1}{2}, 1]$

$K(\omega) = 2\omega$



range of K is $[0, 2]$

No single point with positive prob.

K is a continuous r.v.

C.D.F. (Cumulative distribution function)

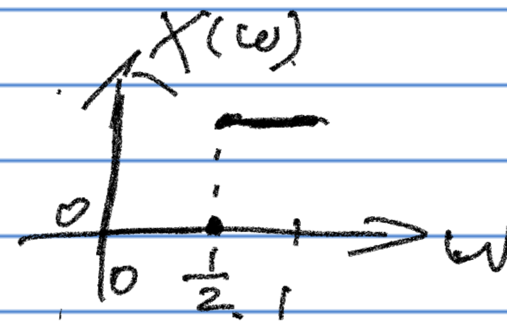
$F(x) = P(X \le x)$, for $x \in \mathbb{R}$

$= P(\{\omega \mid X(\omega) \le x\})$.

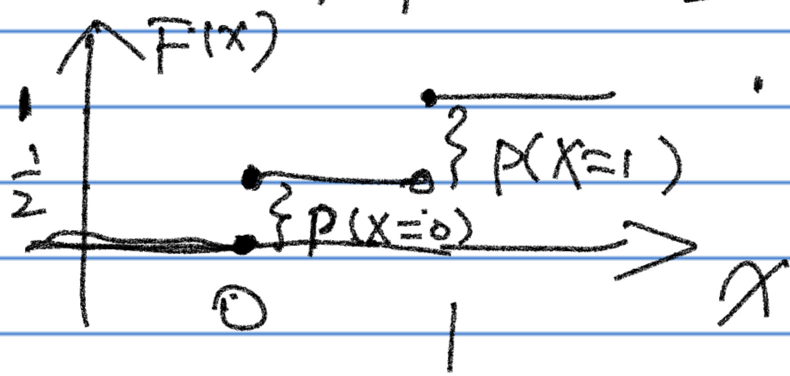
Examples:

$\omega \sim \text{Unif}([0,1])$

1). $X(\omega) = 1(\omega \geq \frac{1}{2})$



$P(X=0) = \frac{1}{2}$, $P(X=1) = \frac{1}{2}$

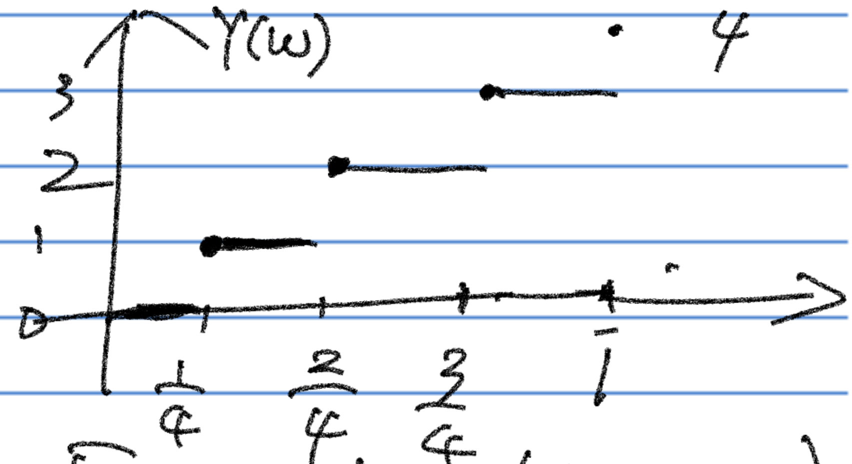


$X \le 0 \iff \omega \in [0, \frac{1}{2}]$

$X \le \frac{3}{4} \iff \omega \in [0, \frac{1}{2}]$

$X \le 1 \iff \omega \in [0, 1]$

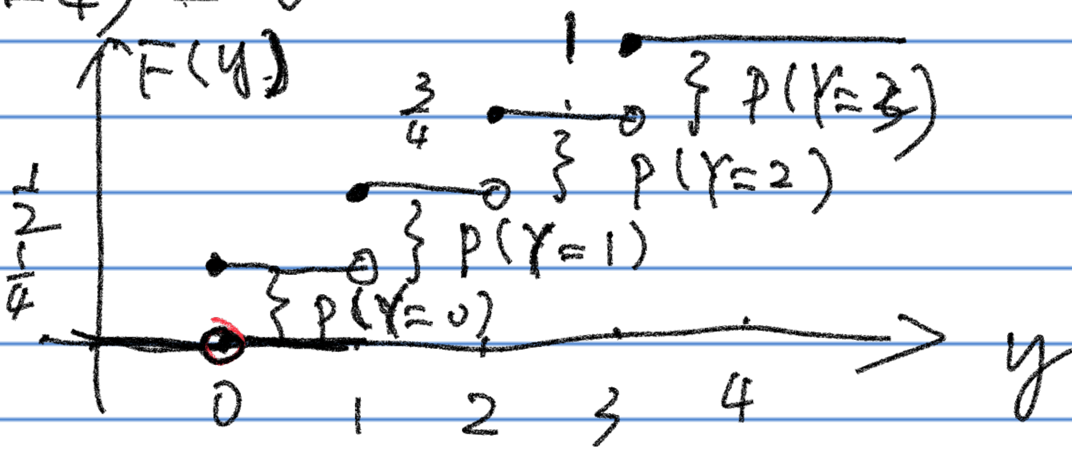
$$Y = \lfloor 4\omega \rfloor$$



$$P(Y=0) = \frac{1}{4}$$
$$P(Y=1) = \frac{1}{4}$$
$$P(Y=2) = \frac{1}{4}$$
$$P(Y=3) = \frac{1}{4}$$
$$P(Y=4) = 0$$

prob Mass Function (P.M.F.)

$$p(x) = P(X = x)$$

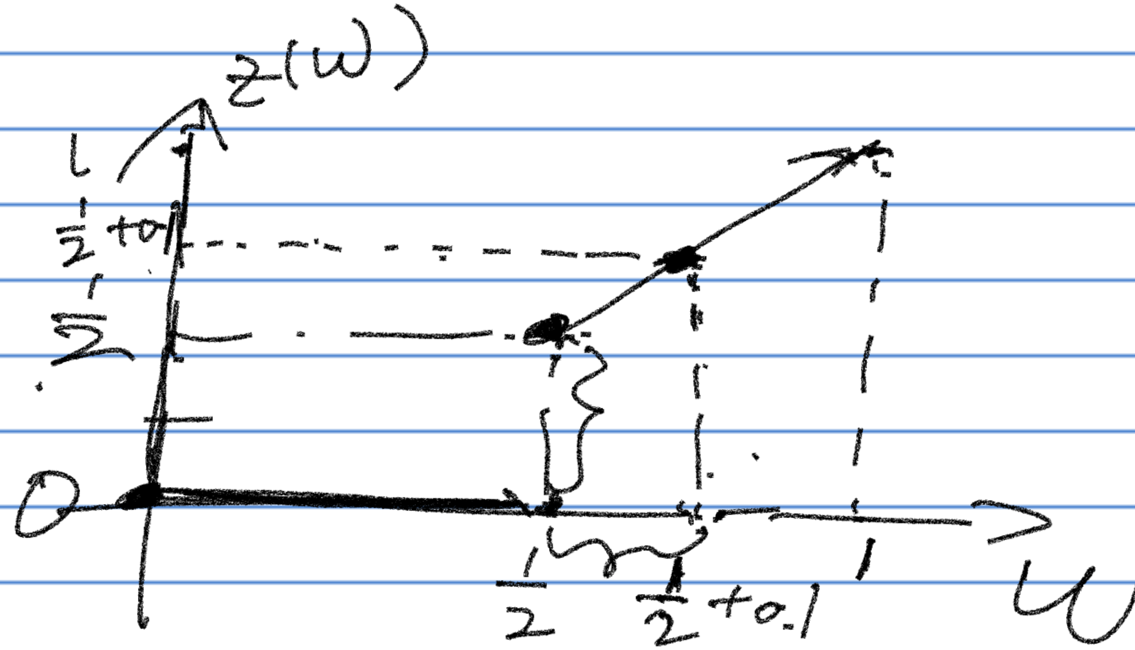


$$Z(\omega) = \begin{cases} 0, & \omega < \frac{1}{2} \\ \omega, & \omega \geq \frac{1}{2} \end{cases}$$

$$P(Z=0) = \frac{1}{2}$$

$$P(Z \leq 0) = P(Z=0)$$

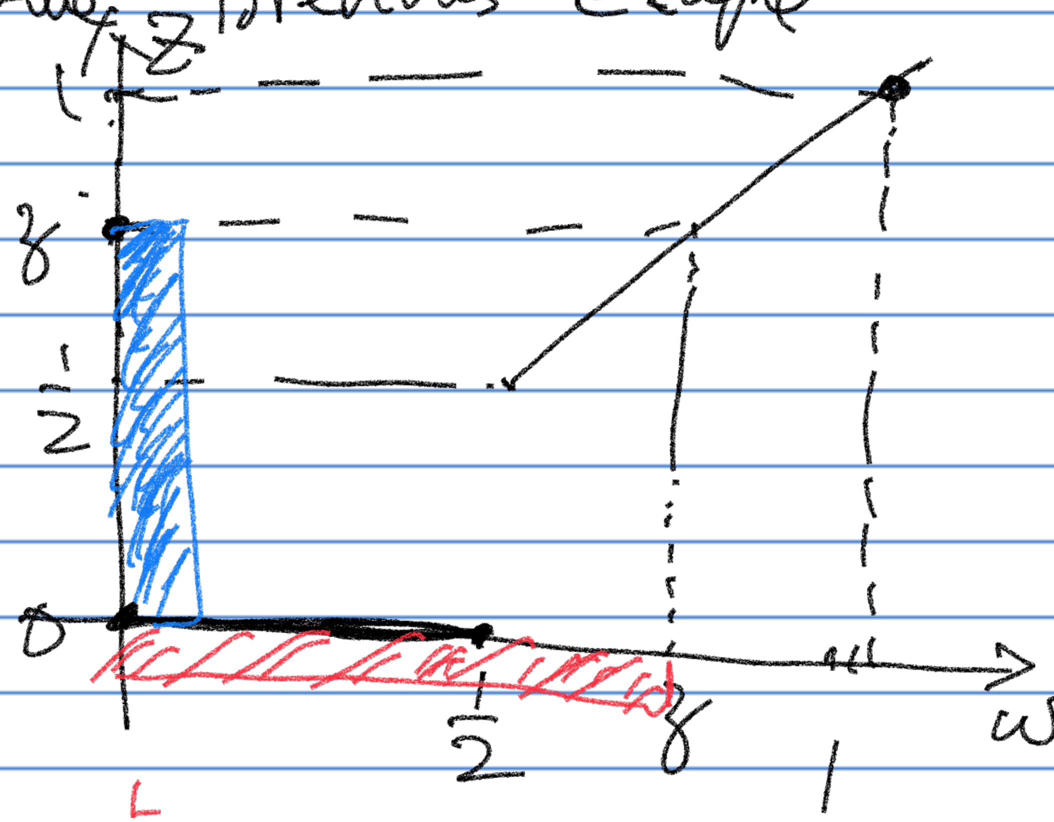
$$P(Z \leq 0.1) = P(Z \leq 0)$$



$$P(Z \leq \tfrac{1}{2} + 0.1) = P(Z=0) + P(\tfrac{1}{2} < Z < \tfrac{1}{2} + 0.1) = \frac{1}{2} + 0.1$$

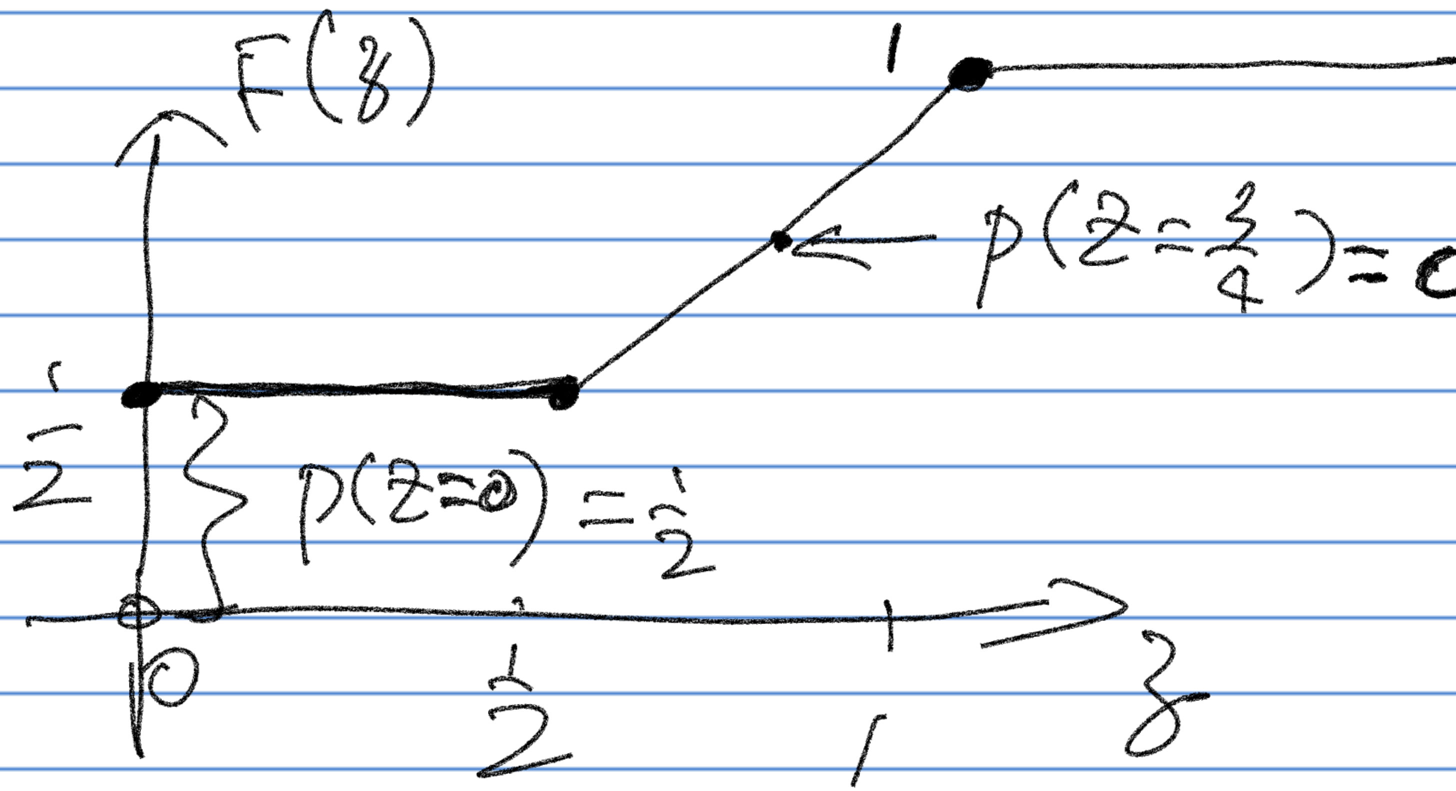
$$Z \leq \frac{1}{2} + 0.1 \iff \{\omega \leq \tfrac{1}{2}\} \cup \{\tfrac{1}{2} \leq \omega < \tfrac{1}{2} + 0.1\}.$$

For the previous example



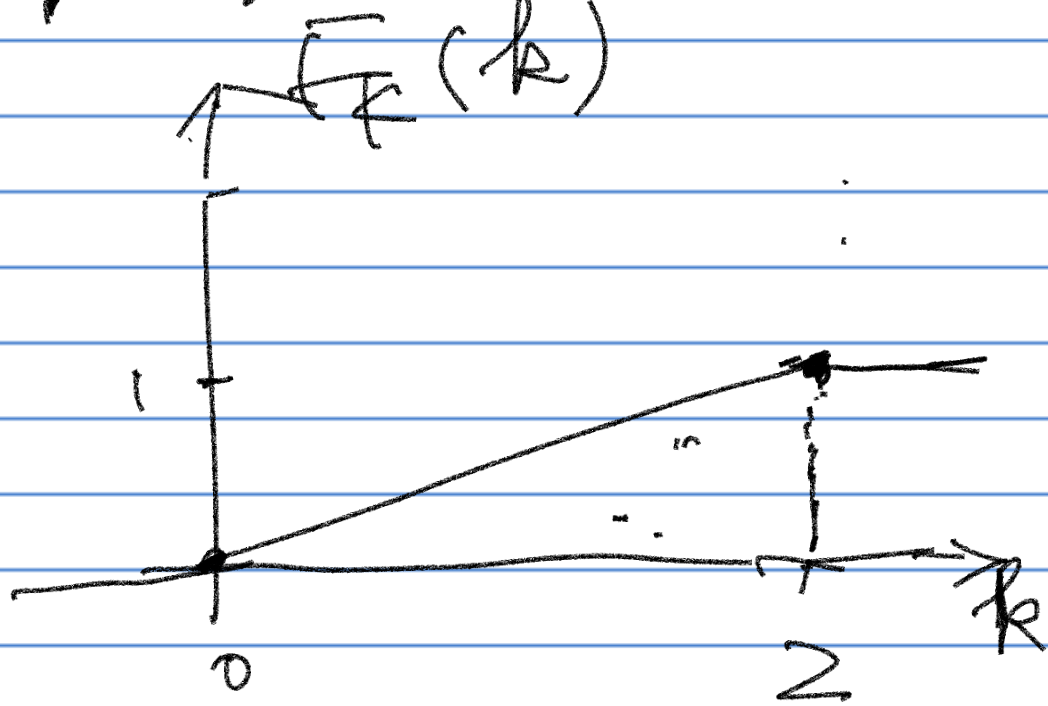
$Z = 0$
$\Leftrightarrow w \in [0, \frac{1}{2}]$

when $\frac{1}{2} \le z \le 1$
$Z \le z$
$\Leftrightarrow w [0, z]$
$P(Z \le z) = z$



F(z)
1
p(z = 3/4) = 0
1/2
p(z=0) = 1/2
0
1/2
1
z

$K(w) = 2w$



$$P(K \le k) = P(2w \le k) = P\left(w \le \frac{k}{2}\right) = \frac{k}{2}.$$

$$P(K = k) = 0.$$

$$F_K(k) = \frac{k}{2}.$$