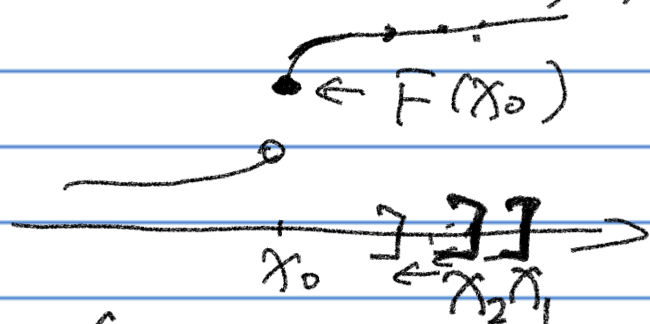
Lecture 6

Longhai Li, September 23, 2021

Properties if C.D.F.

$$F(x) = P(X \le x)$$

1. $F(a) \le F(b)$ if $a \le b$, non-decreasingly
2. $\lim\limits_{x \to +\infty} F(x) = 1$, $\lim\limits_{x \to -\infty} F(x) = 0$
3. $\lim\limits_{x \downarrow x_0} F(x) = F(x_0)$, right continuous.



$$x_1 > x_2 > \cdots\cdots x_0 \qquad \lim_{i \to +\infty} x_i = x_0$$

$$\{X \le x_1\} \supseteq \{X \le x_2\} \supseteq \cdots\cdots \downarrow$$

$$\{X \le x_i\} \downarrow \underline{\{X \le x_0\}} = \underline{\bigcap_{i=1}^{+\infty} \{X \le x_i\}}$$

$$\left\{ \begin{array}{l} X(\omega) \le x_0 \Rightarrow X(\omega) \le x_i \text{ for all } i \\ X(\omega) \le x_i \text{ for all } i \Rightarrow X(\omega) \le \lim_{i \to +\infty} x_i = x_0 \end{array} \right.$$

By continuity of prob.

$$P\left(\lim_{i \to +\infty} \{X \le x_i\}\right) = \lim_{i \to +\infty} P(X \le x_i)$$

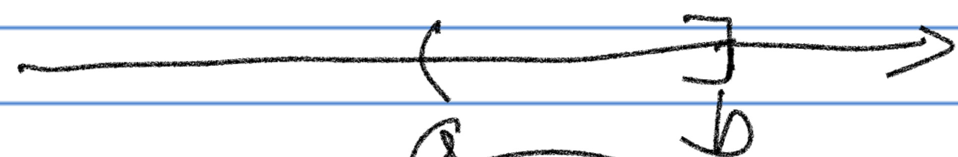
$$F(x_0) = P(X \le x_0) = \lim_{i \to +\infty} F(x_i)$$

4. $F(b) - F(a) = P(a < X \le b)$ for $a < b$

Pf: $$P(X \le b) - P(X \le a) = P(a < X \le b)$$

$$\Leftrightarrow P(X \le b) = P(X \le a) + P(a < X \le b)$$

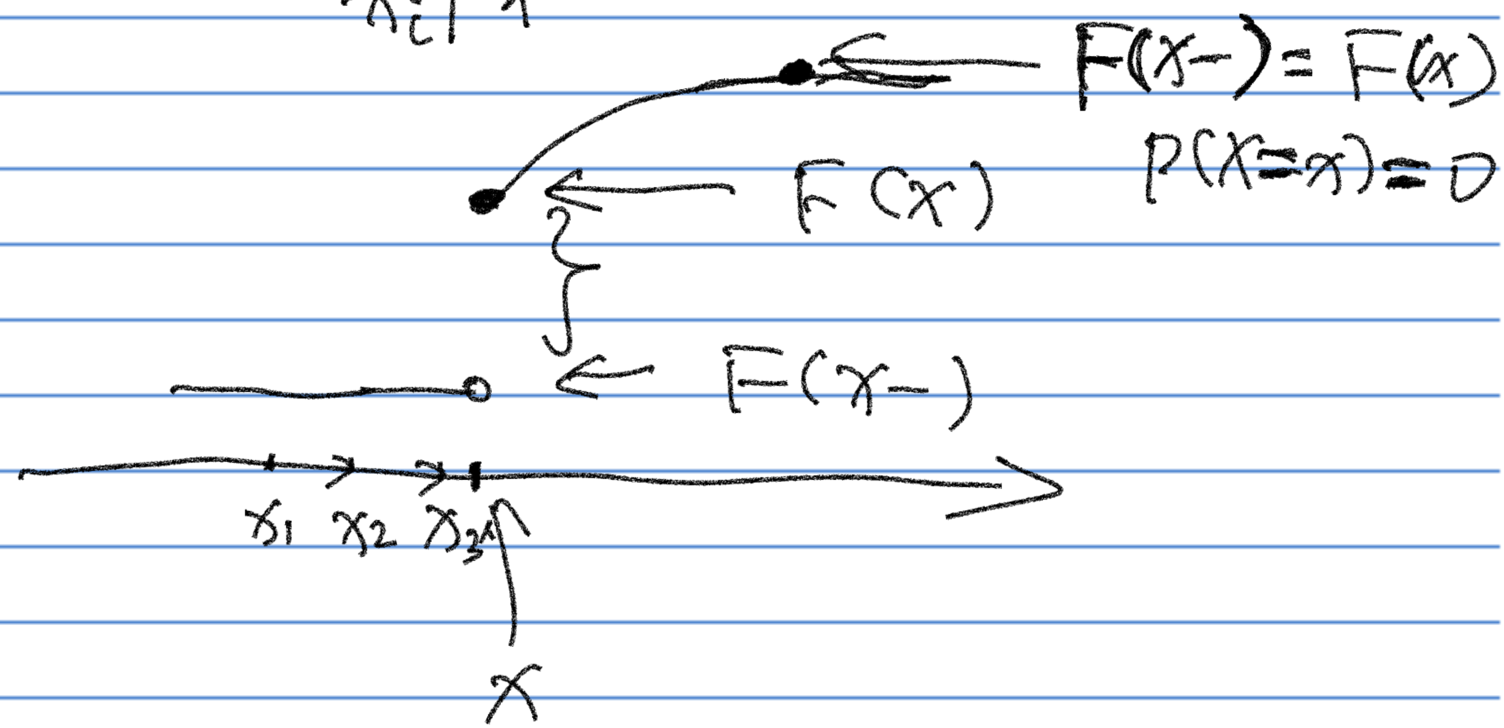
$X \le a$]   ] $\leftarrow X \le b$



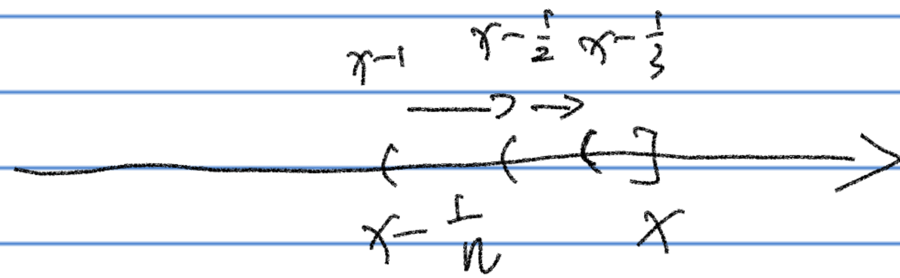
$$X \le a \;\cup\; X \in (a, b] = X \in (-\infty, b]$$

5. $F(x) - F(x-) = P(X=x)$

$$F(x-) = \lim_{x_i \uparrow x} F(x_i)$$



例: $\{X = x\} = \lim\limits_{n\to+\infty}\{\omega \mid X(\omega)\in(x-\frac{1}{n}, x]\}$



$$x-\frac{1}{n} \uparrow x$$

$$= \bigcap_{n=1}^{+\infty}\{X\in(x-\tfrac{1}{n}, x]\}$$

$$P(X=x) = \lim_{n\to+\infty} P(x-\tfrac{1}{n} < X \le x).$$

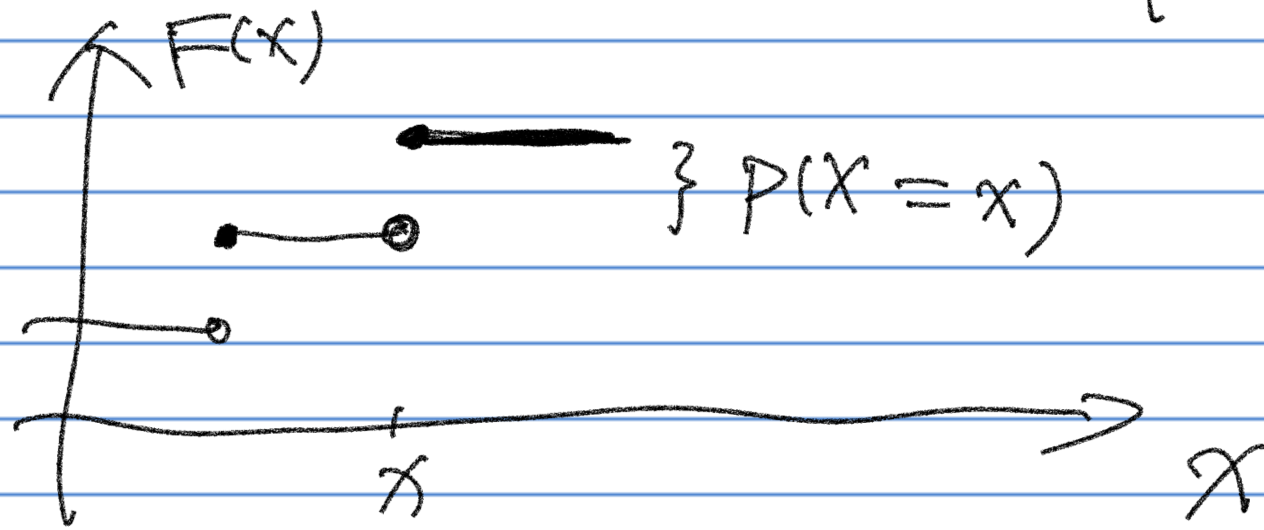
$$= \lim_{n\to+\infty}\left[F(x)-F(x-\tfrac{1}{n})\right]$$

$$= F(x) - F(x-)$$

Discrete R.V.

Def: Range of X is a countable set

C.D.F. of X is a step function



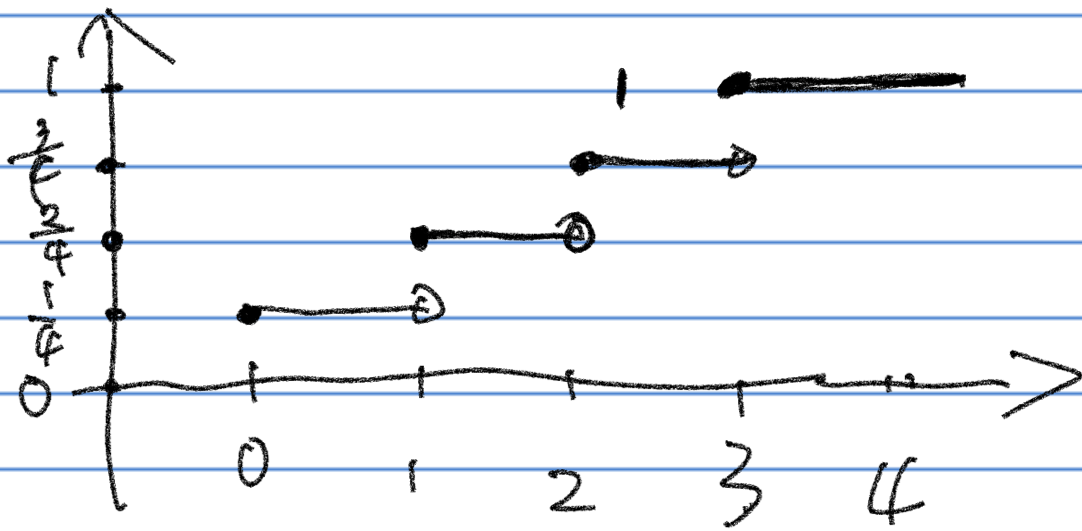
Probability Mass Function (P.M.F.)

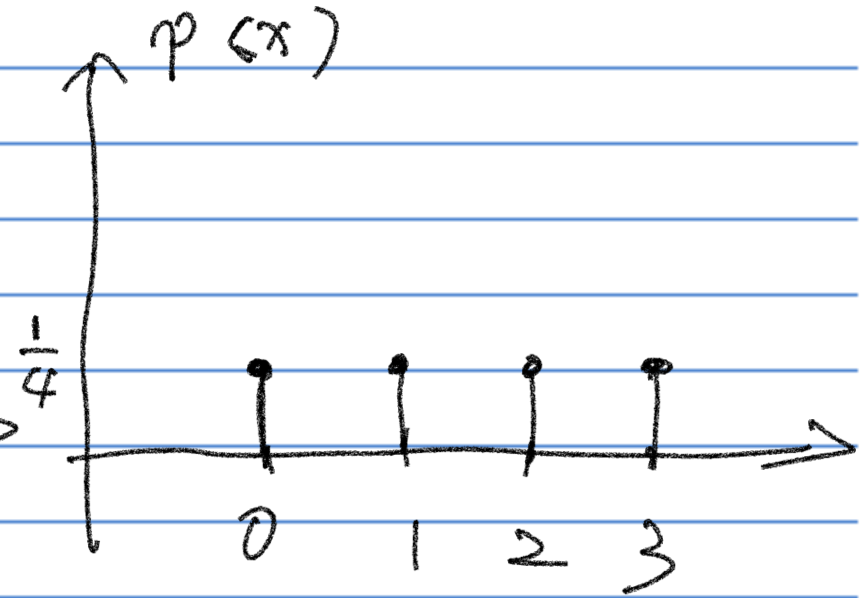
$$p(x) = P(X = x) \quad \text{for all } x \in \text{Range}(X)$$

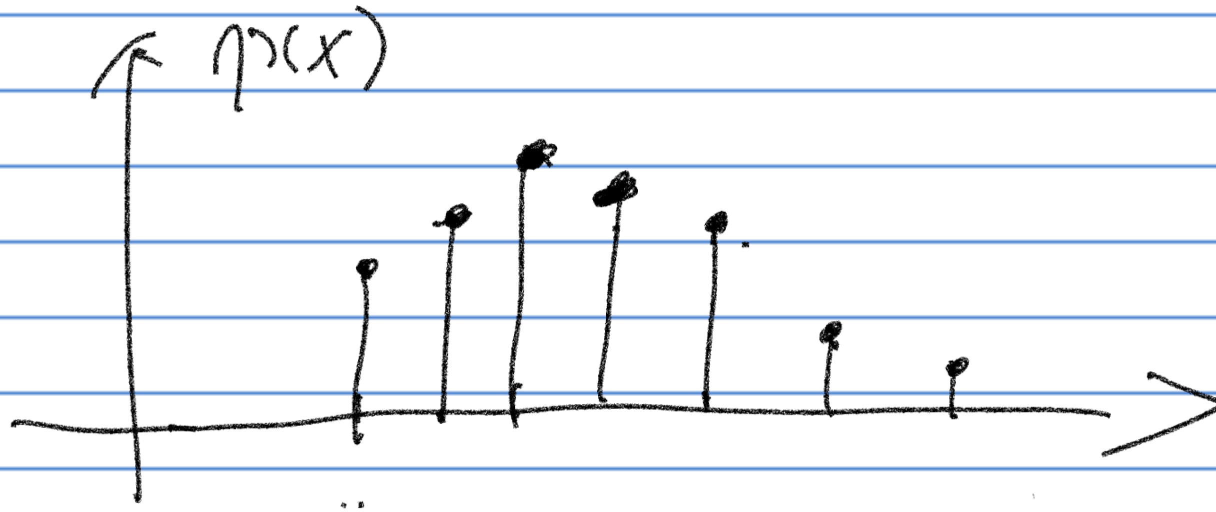
$$= F(x) - F(x-)$$

Example

$$X = \lfloor 4w \rfloor, \quad w \sim \mathrm{Unif}([0,1])$$







$$\begin{cases} p(x) \geq 0 \\ \displaystyle\sum_{x \in \mathrm{Range}(X)} p(x) = 1 \end{cases}$$

Transformation of a discrete r.v.

$$Y = g(X)$$

$$p_Y(y) = P(Y = y)$$

$$= P(g(X) = y)$$

$$= \sum_{g(x)=y} p_X(x)$$

$$\{g(X) = y\} = \bigcup_{g(x)=y} \{X = x\}$$

Example:

X has the following p.m.f.

x	-1	0	1
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$Y = X^2$

y	0	1
$p_Y(y)$	$\frac{1}{2}$	$\frac{1}{2}$

$P(Y=0) = P(X=0)$

$P(Y=1)$
$= P(X=-1)$
$+ P(X=1)$
$= \frac{1}{2}$

$Z = 2X$ (1-1 transformation)

z	-2	0	2
$p_Z(z)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$P(Z = -2)$
$= P(X = -1)$

Continuous R.V.

Def: The C.D.F. of X is continuous.

or $P(X=x)=0$ for all $x \in \mathbb{R}$

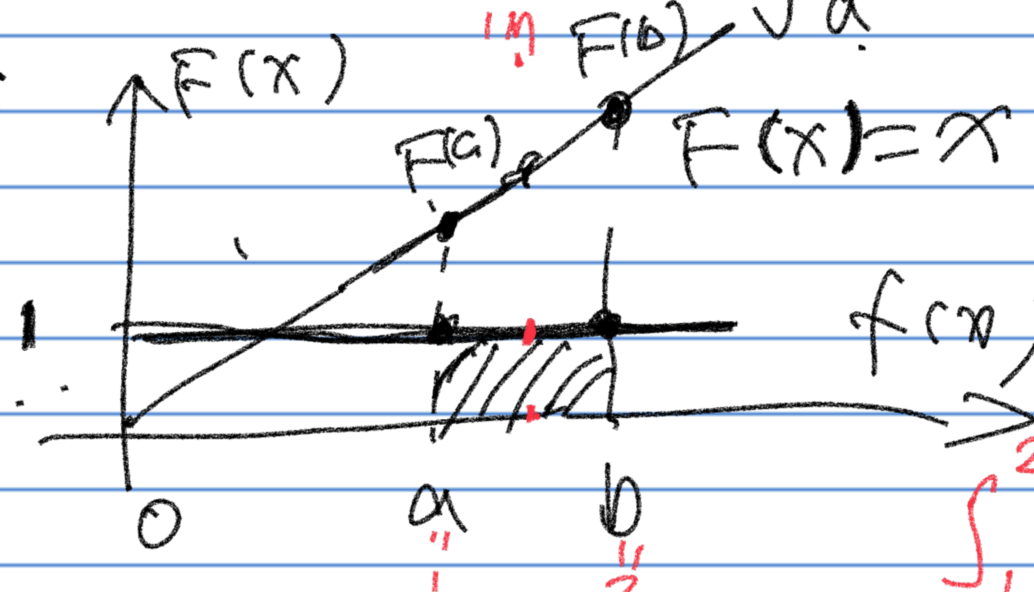
$\vdots$

Fundamental Theorem of Calculus

If F is a continuous function $F'(x) = f(x)$ ~~for~~ almost all $x \in \mathbb{R}$.

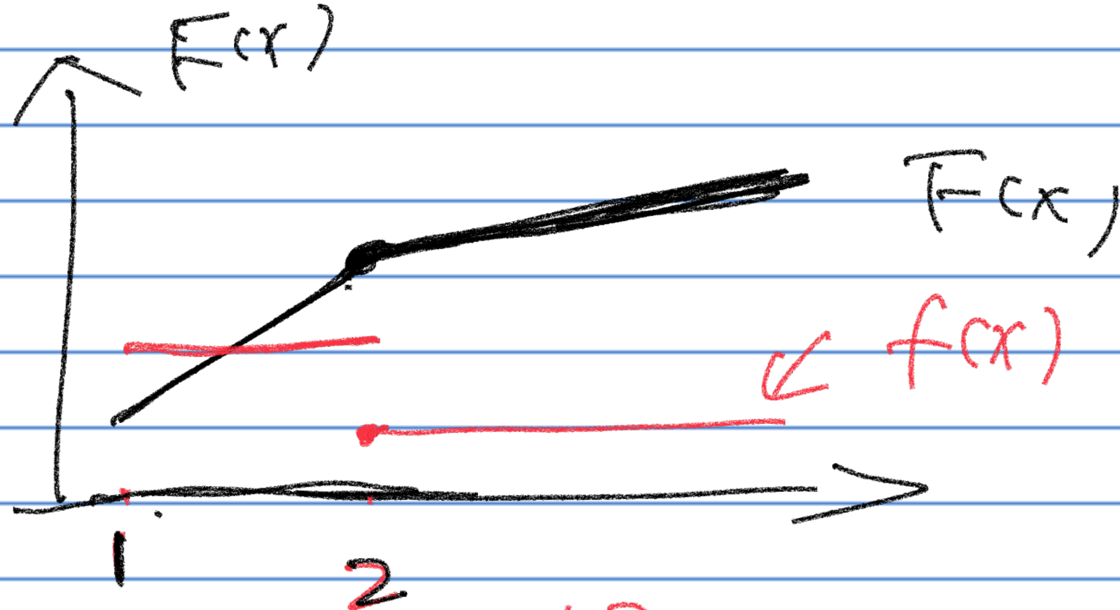
then $F(b) - F(a) = \int_a^b f(x)\,dx$ ✓

Example:



$\int_1^2 f(x) = F(2) - F(1)$

$f(1.5) = 1M.$

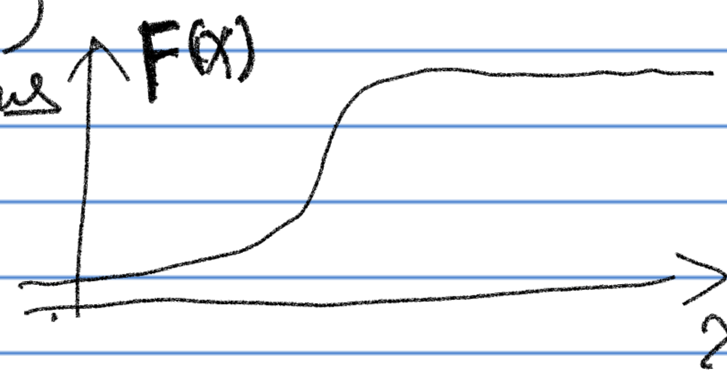


F is NOT diff at 2.

$F'(x) = f(x)$ for all

almost all $x \in \mathbb{R}$

Prob. density function. (P.D.F.)



$f(x)$ is a P.D.F of a continuous X
if

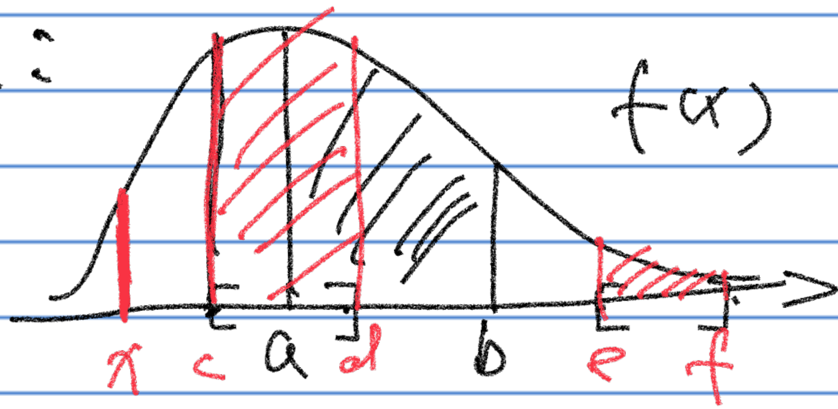
Def1: $f(x) = F'(x)$ when $F'(x)$ exists.

Def2: $F(x) = \int_{-\infty}^{x} f(t)\,dt$, for all $x \in \mathbb{R}$.

Def3: $F(b) - F(a) = \int_a^b f(x)\,dx$

for all $a, b \in \mathbb{R}$

Example:



$$\int_a^b f(x)\,dx = P(a \le X \le b)$$

$$= F(b) - F(a) \quad \text{for all } a, b$$

$$P(X \in (c,d)) > P(X \in (e,f))$$

$$P(X = x) = 0$$