Lecture 7

Longhai Li, September 28, 2021

Def of P.D.F.

X is an absolutely continuous R.V. $(P(X=x)=0)$

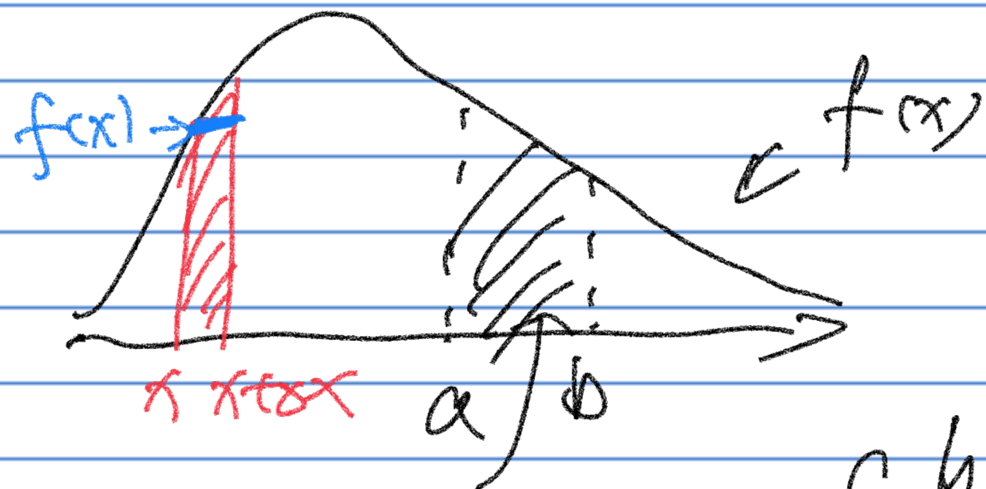
$f(x)$ is a p.D.F. of X if

(2) $F(x)=\int_{-\infty}^{x} f(t)\,dt$, for all x

or

(3) $P(a \leq X \leq b)=\int_a^b f(x)\,dx$, for all a, b.

(1) $f(x)=F'(x)$ for almost all $x \in \mathbb{R}$



$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

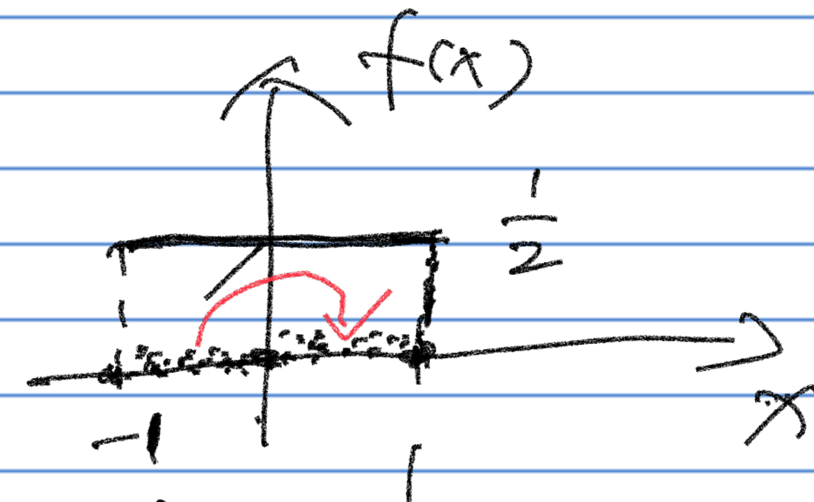
$$P(x \le X \le x+\Delta x) \approx f(x)\cdot \Delta x$$

$$f(x) \approx \frac{P(x \le X \le x+\Delta x)}{\Delta x}$$

Transformation of R.V.

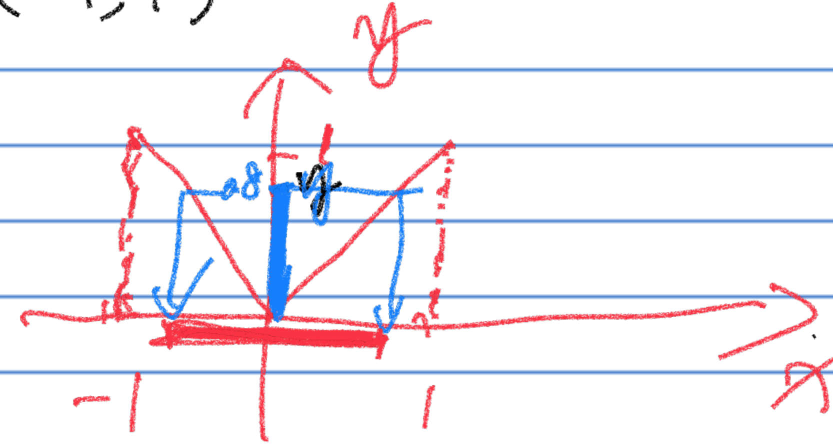
Example:

$X \sim \text{Unif}((-1,1))$



$f(x) = \frac{1}{2}$, for $x \in (-1,1)$

$Y = |X|$



Find the C.D.F. of Y:

For $y \in [0,1)$

$$F_Y(y) = P(Y \le y)$$

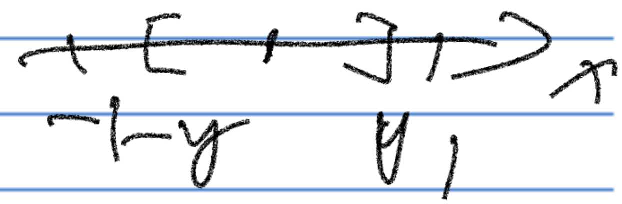
$$= P(|X| \le y)$$

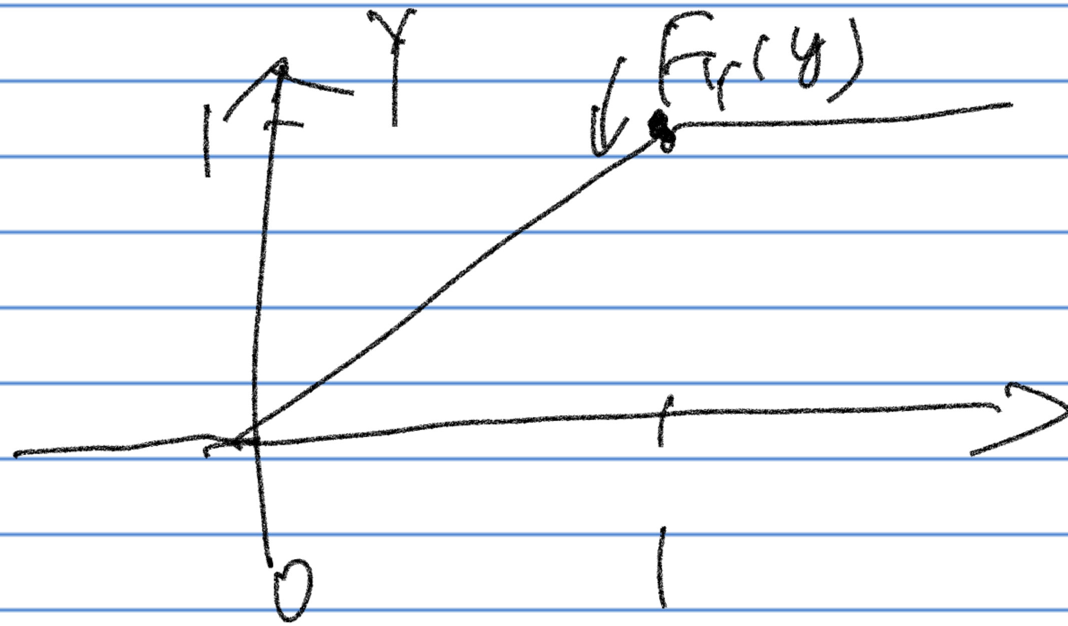
$$= P(-y \le X \le y)$$

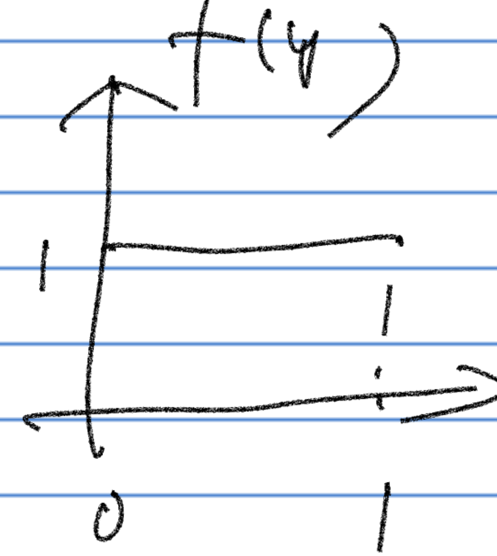
$$= F_X(y) - F_X(-y)$$

$$= \frac{2y}{2} = y$$

$$P(X \in I) = \frac{\text{length}(I)}{2}$$



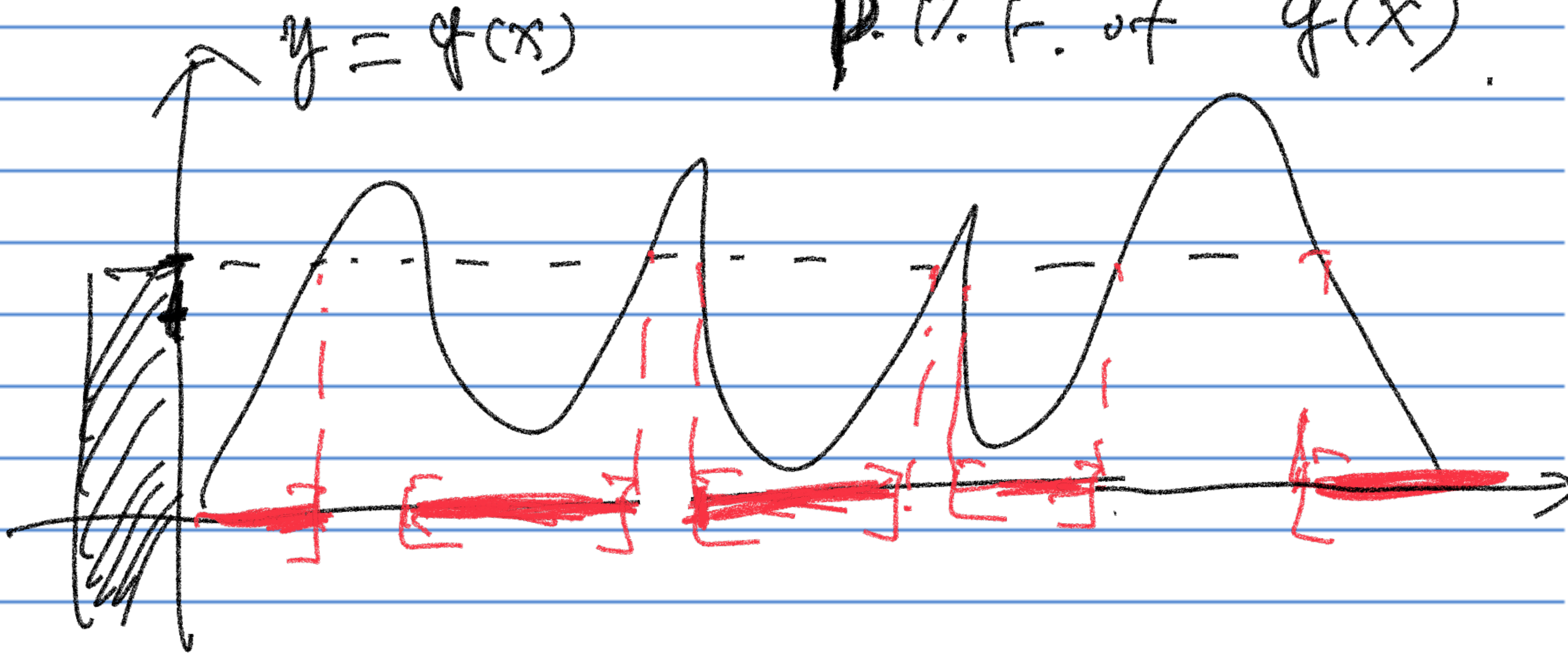




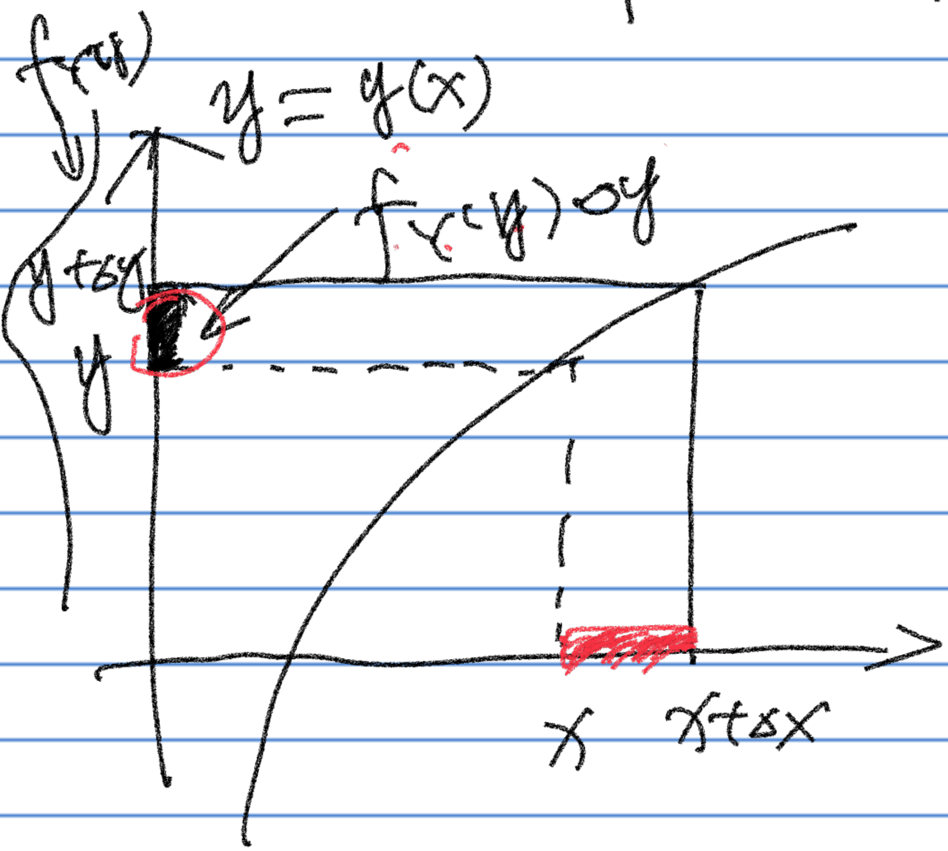
$\therefore f_Y(y) = F_Y'(y) = 1$

for $y \in [0, 1)$

Generally, it is diff to find the p.d.f. of $g(X)$.



1-1 transfer. of X.



X has a P.D.F. $f_X(x)$. Let's use $f_Y(y)$ to denote the P.D.F. of Y.

$$f_Y(y)\cdot\Delta y = f_X(x)\cdot\Delta x$$

$$f_Y(y) = f_X(x)\cdot\frac{\Delta x}{\Delta y}$$

$g^{-1}(y)$ exists

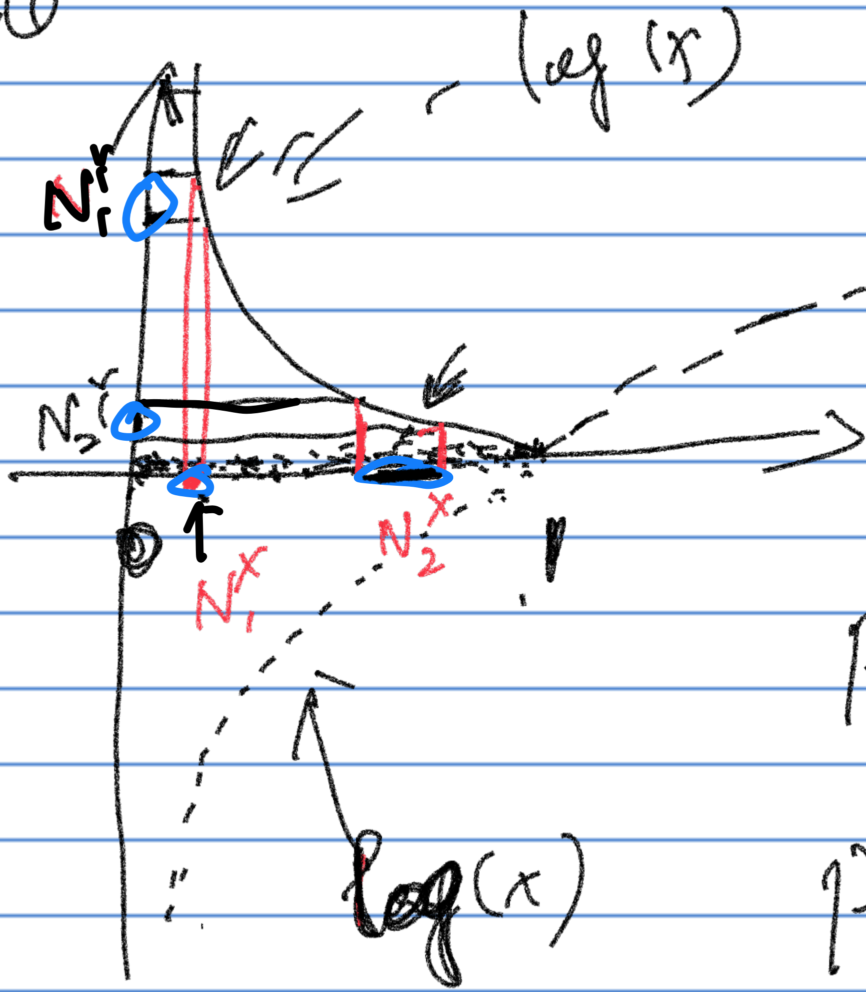
$$f_Y(y) = f_X(x) \cdot \frac{\Delta x}{\Delta y}$$

$$= f_X(x) \left| \frac{d}{dy} g^{-1}(y) \right| \Leftarrow$$

$$f_Y(y) |dy| = f_X(x) |dx|$$

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Example.



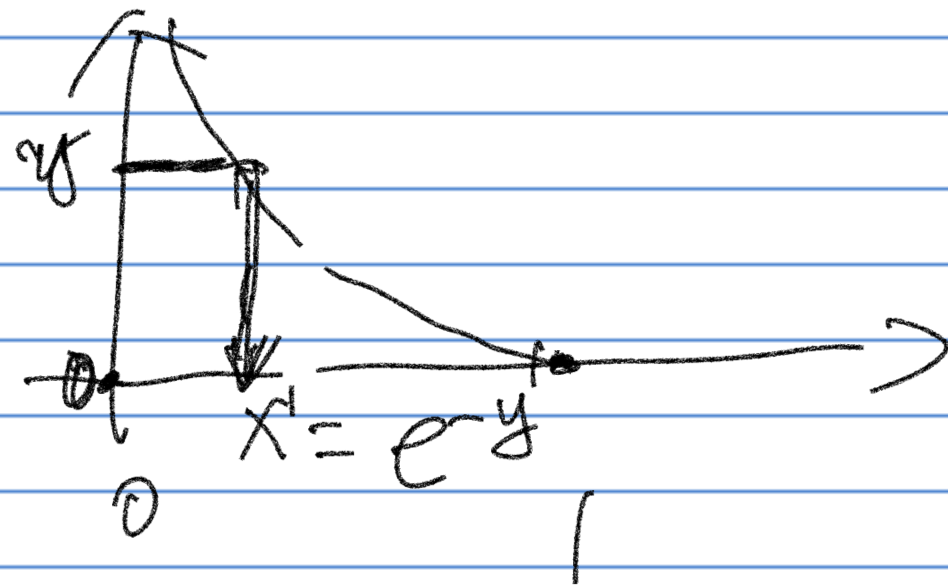
$X \sim \mathrm{Unif}([0,1])$

$Y = -\log(x)$

$P(Y \in N_1^Y)$

$\wedge$

$P(Y \in N_2^Y)$.



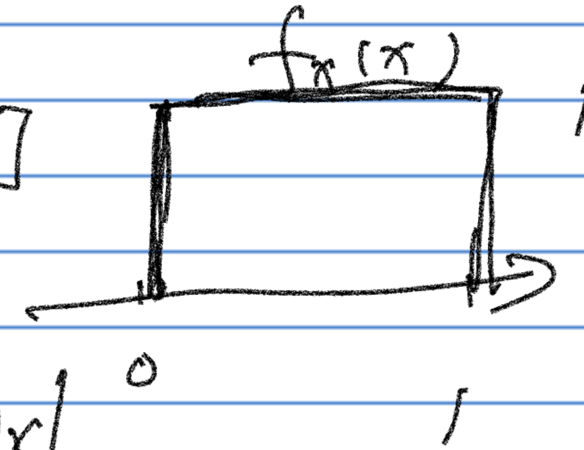
$y = -\log(x)$

$x = e^{-y}$

Jacobian

$$\left|\frac{dx}{dy}\right| = \left|e^{-y}\cdot(-1)\right| = e^{-y}$$

$$f_X(x) = 1, \text{ for } x \in [0,1]$$



$$f_Y(y) \cdot |dy| = f_X(x)\,|dx|$$

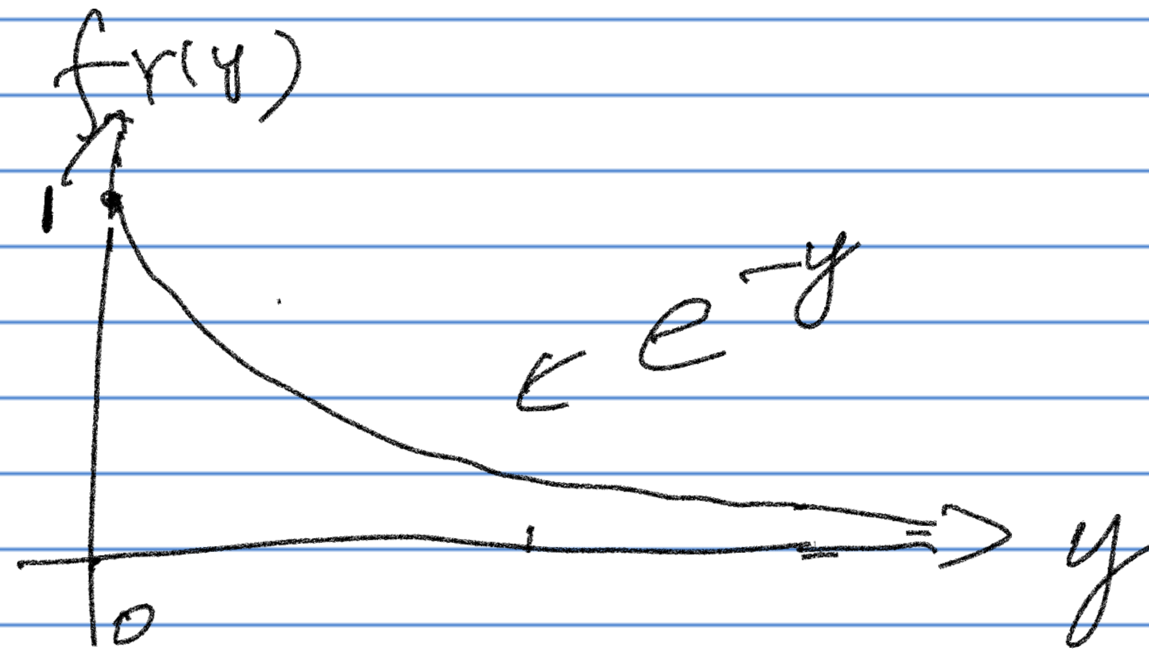
For $y > 0$

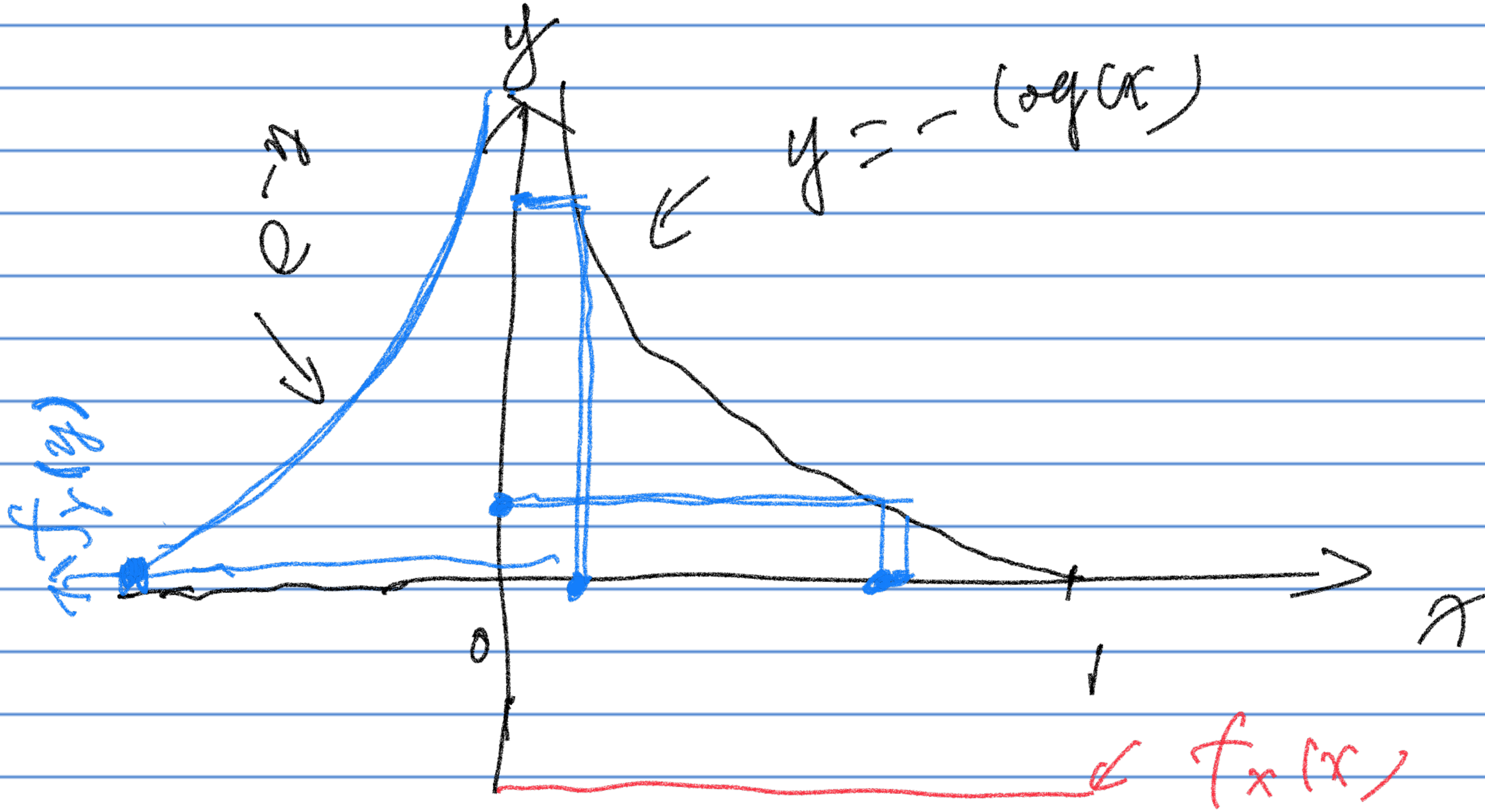
$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$$

$e^{-y} \in [0,1]$

$$= f_X(e^{-y})\, e^{-y}$$

$$= 1 \times e^{-y}.$$


$f_Y(y)$
1
0
e^{-y}
y

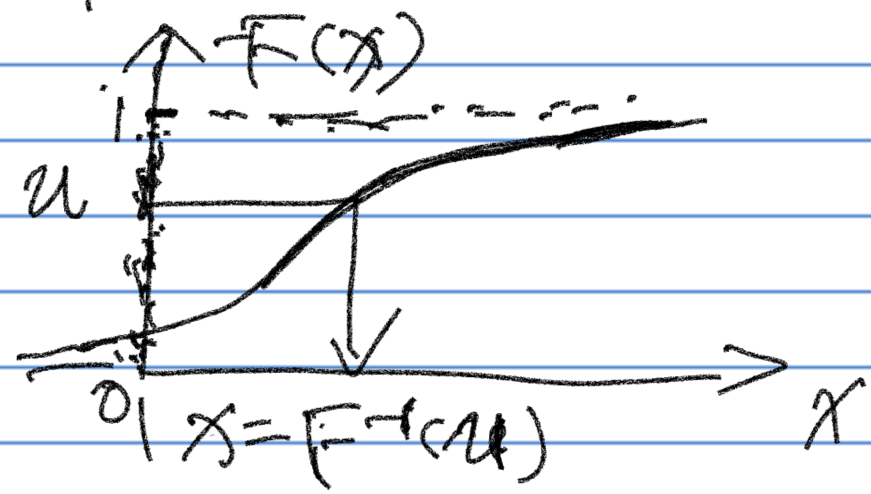


y
y = - log(x)
e^{-y}
f_Y(y)
0
1
x
f_X(x)

Thm:

X is a continous R.V. with a C.D.F. $F(x)$

$F(x)$ is a strictly monotone, i.e. $F^{-1}(x)$ exists



$U \sim \text{Unif}([0,1])$

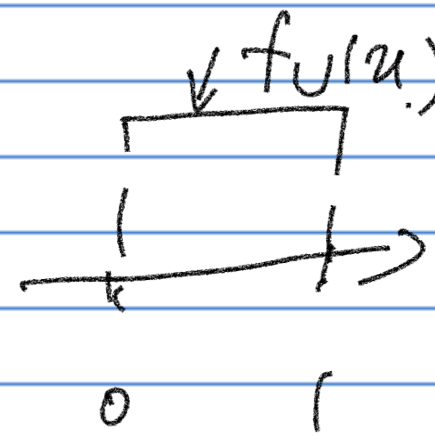
Then the C.D.F. of $F^{-1}(U)$ is $F(x)$

pf:

$X = F^{-1}(U)$

$U = F(X)$

$U \rightleftarrows X$

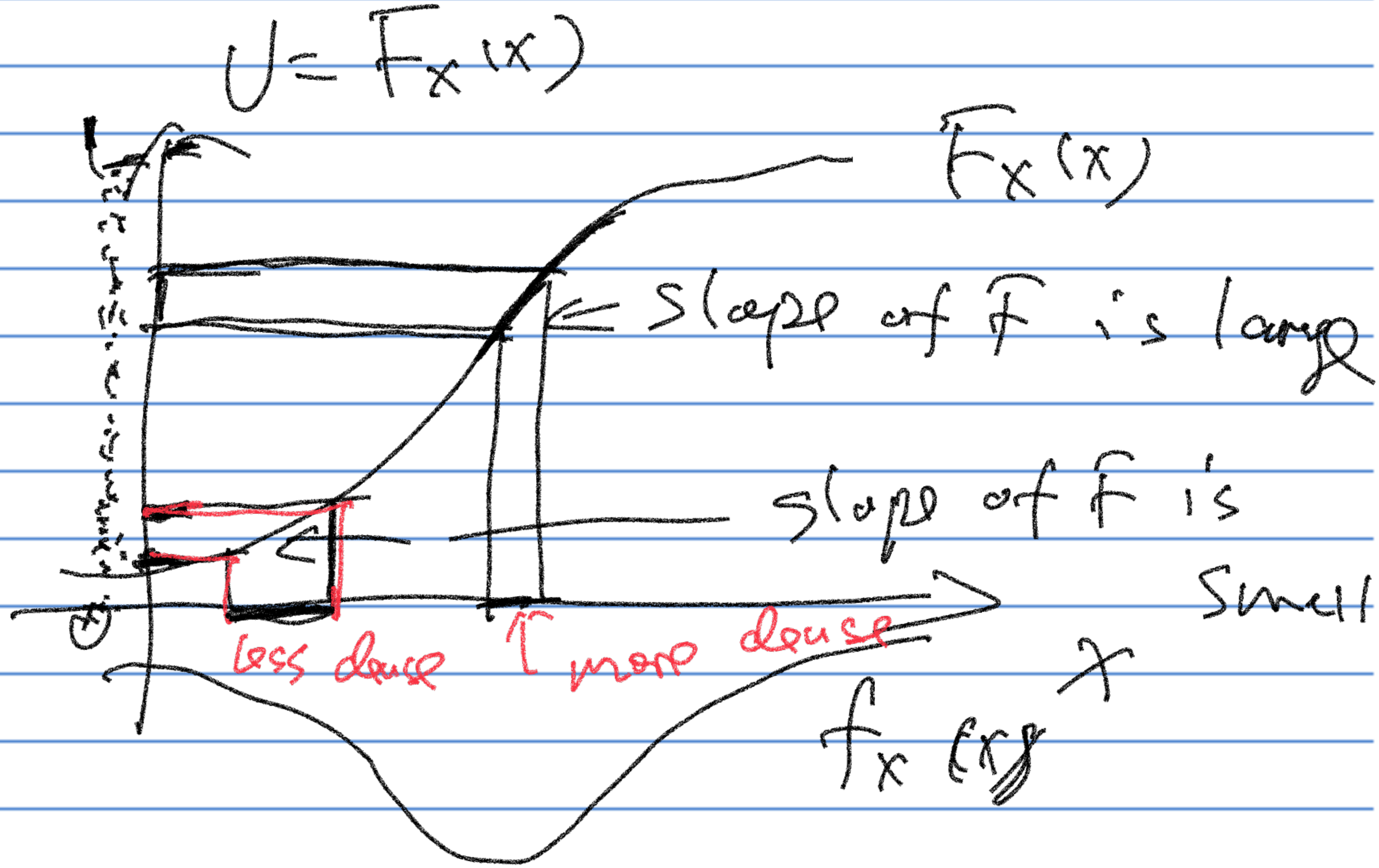


$f(x) = F'(x)$

$$\left|\frac{du}{dx}\right| = F'(x) = f(x)$$

$$f_X(x)\,|dx| = f_U(u)\,|du|$$

$$f_X(x) = f_U(F(x)) \cdot f(x) = f(x)$$


U = Fx (x)
Fx (x)
slope of F is large
slope of F is small
less dense
more dense
fx (x)

Expectation of a r.v.

Def of $\mathbb{E}(X)$ & $E(g(X))$.

(1) X is discrete with a p.m.f. $p(x)$

$$E(X) = \sum_{\text{all possible } x} x \cdot p(x)$$

$$E(g(X)) = \sum_{\text{all possible } x} g(x)\, p(x)$$

$Y = g(X)$

$$E(Y) = \sum_{\text{all possible } y} y \cdot p_Y(y)$$

(2) X is a continuous R.V. with a P.D.F. $f(x)$

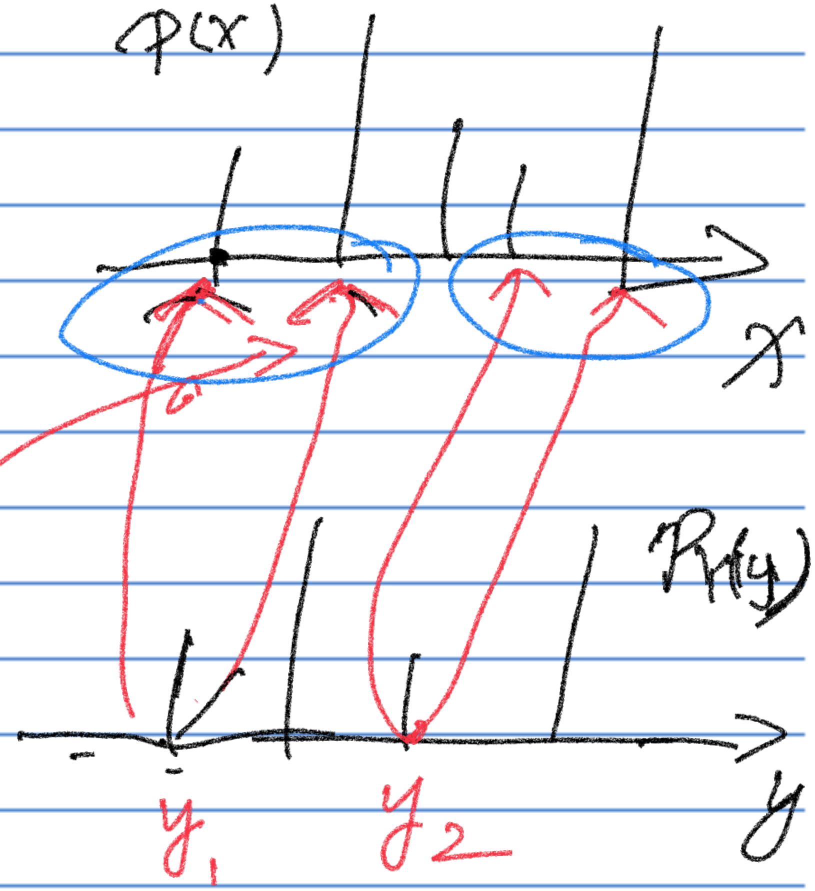
$$E(X) = \int_{-\infty}^{+\infty} x \, f(x) \, dx$$

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) \, f(x) \, dx$$

Explanation of $\mathbb{E}(g(X))$ for discrete case.

$$Y = g(X),$$

$$p_Y(y) = \sum_{\{x \mid g(x)=y\}} p(x)$$



$$E(Y) = \sum_{\text{all } y} y \cdot p_Y(y)$$

$$= \sum_{\text{all } y} y \cdot \sum_{\{x \mid g(x)=y\}} p(x)$$

$$= \sum_{\text{all } y} \sum_{\{x \mid g(x)=y\}} y \cdot p(x)$$

$$= \sum_{\text{all } y} \sum_{\{x \mid g(x)=y\}} g(x) \cdot p(x)$$

$$= \sum_{\text{all } x} g(x) \cdot p(x)$$

Example

$$X \sim \begin{array}{c|ccc} x & -1 & 0 & 1 \\ \hline P_X(x) & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array}$$

$$Y = |X|$$

$$Y \sim \begin{array}{c|cc} y & 0 & 1 \\ \hline P_Y(y) & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$0 \times \frac{1}{2} + 1 \times \frac{1}{2} = 0 \times \frac{1}{2} + \left(|-1| \cdot \frac{1}{4} + |1| \cdot \frac{1}{4} \right)$$