Lecture 8

Longhai Li, October 5, 2021

Def of $E(X)$:

1. X is discrete.

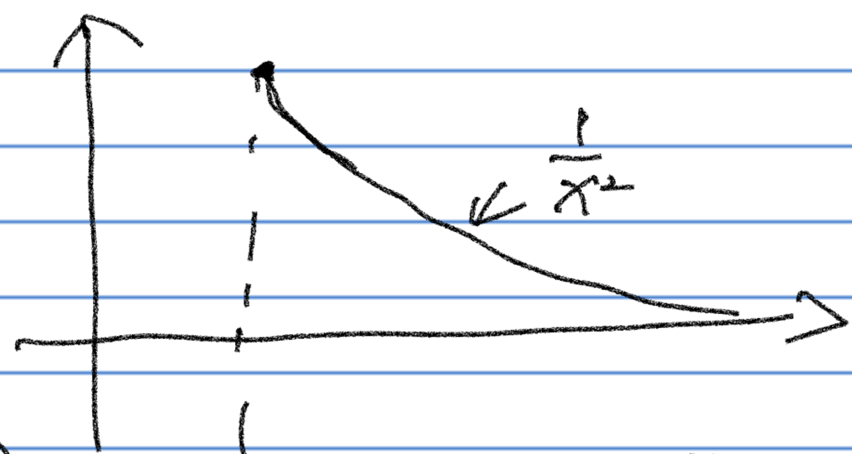
$$E(g(X)) = \sum_{\text{all possible } x} g(x) \cdot p(x)$$

2. X is continuous

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) f(x) \, dx$$

Example of non-finite Expectation:

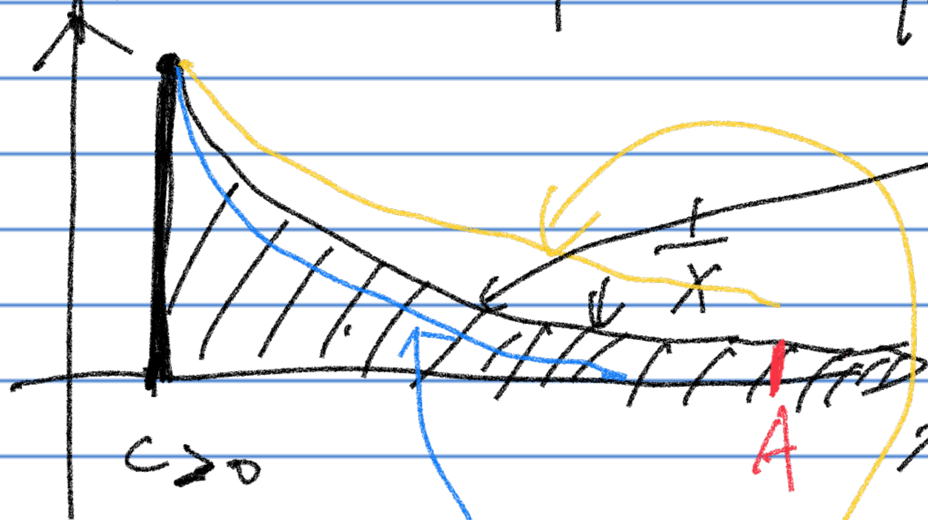
$f(x) = \frac{1}{x^2}$, for $x \geq 1$



$$\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{+\infty}$$

$$= 0 + 1 = 1$$

Integrals of power functions



$$\int_c^{+\infty} \frac{1}{x} = +\infty$$

$$\int_c^{+\infty} \frac{1}{x^{1+\delta}} < +\infty$$

for $\delta > 0$

$$\int_c^{+\infty} \frac{1}{x^{1-\delta}} = +\infty$$

for $\delta \geq 0$

$$f(x) = \frac{1}{x^2}, \quad \text{for} \quad x \geq 1$$

$$E(X) = \int_1^{+\infty} x \cdot f(x)\, dx$$

$$= \int_1^{+\infty} x \cdot \frac{1}{x^2}\, dx$$

$$= \int_1^{+\infty} \frac{1}{x}\, dx = +\infty.$$

$$= \log x \Big|_1^{+\infty} = +\infty - 0 = +\infty$$

Linearity of E.

$$E\left(k_1 g_1(X) + k_2 g_2(X)\right)$$

$$= k_1 E(g_1(X)) + k_2 E(g_2(X)).$$

Pf:

Suppose X is continuous.

$$LHS = \int_{-\infty}^{+\infty} \left(k_1 g_1(x) + k_2 g_2(x) \right) dx$$

$$= k_1 \int_{-\infty}^{+\infty} g_1(x)\,dx + k_2 \int_{-\infty}^{+\infty} g_2(x)\,dx$$

$$= k_1 E\left(g_1(X) \right) + k_2 E\left(g_2(X) \right)$$

Thm: $X \geq 0$ is a r.v. $F(x)$ is the C.D.F of X. Then

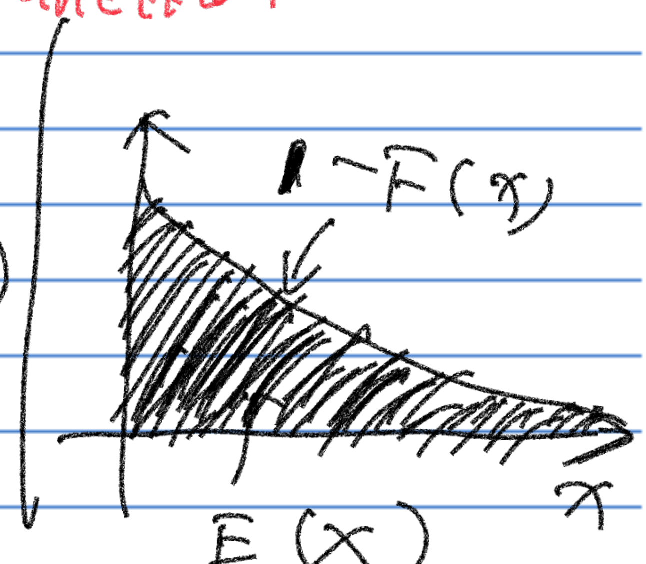
$$E(X) = \int_0^{+\infty} \underbrace{[1 - F(x)]}_{\text{survival function}}\, dx$$

General. $X^+ = \max(X, 0)$

$X^- = \max(-X, 0)$

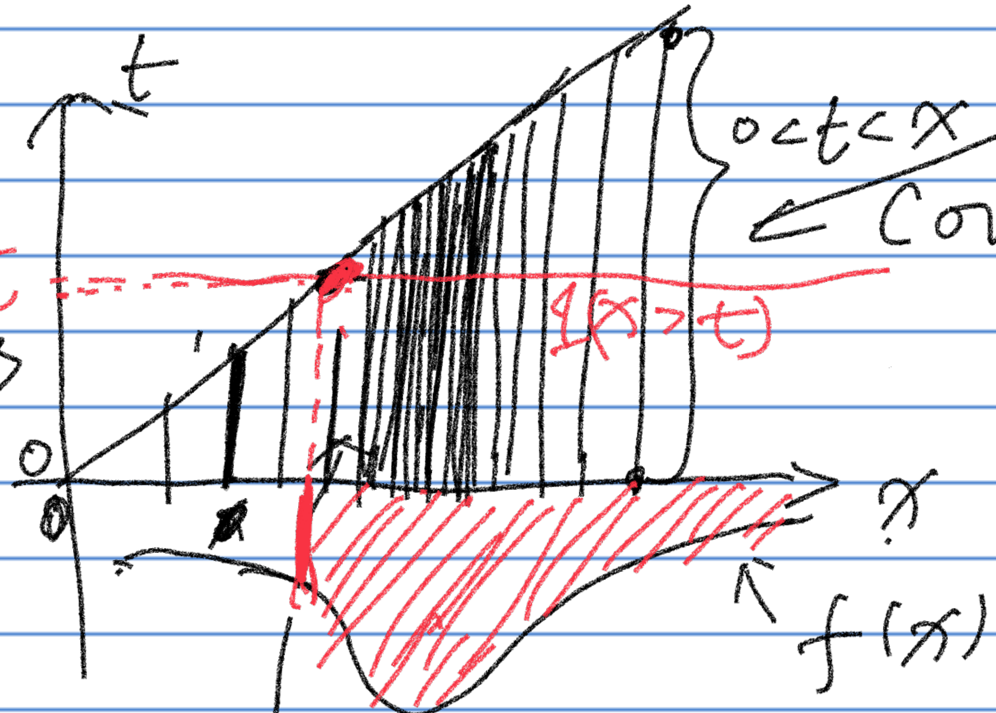
$X = X^+ - X^-$

$E(X) = E(X^+) - E(X^-)$



proof:

Assume X is continuous



$1(0<t<x)$ Contour of $g(x,t)$

$$\underline{g(x,t)} = f(x)\cdot\boxed{1(0<t<x)}$$

$$\underbrace{\int_{-\infty}^{+\infty} x f(x)\,dx}_{= E(X)} = \int_{-\infty}^{+\infty} f(x) \left(\int_0^x 1\, dt \right) dx$$

$$= \int_{-\infty}^{+\infty} f(x) \int_{-\infty}^{+\infty} 1(0<t<x)\, dt\, dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x)\, 1(0<t<x)\, dt\, dx$$

Fubini theorem.

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x)\, 1(0<t<x)\, dx\, dt$$

$$= \int_{-\infty}^{+\infty} \int_{t}^{+\infty} f(x)\, dx\, dt$$

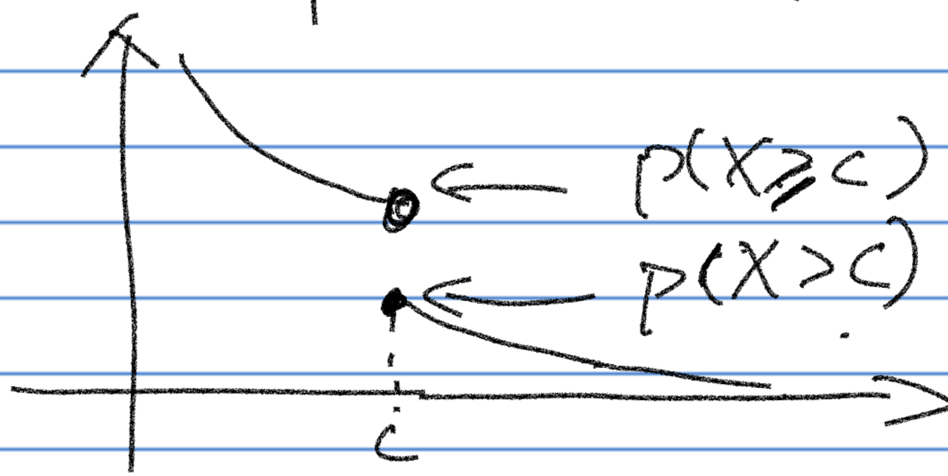
$$= \int_{-\infty}^{+\infty} [1 - F(t)]\, dt$$

A note:

$$E(X) = \int_0^{+\infty} p(X>t)\,dt$$

$$= \int_0^{+\infty} p(X\geq t)\,dt$$

$p(X>t) = p(X \geq t)$ for almost everywhere



The integrals under $p(X>t)$ & $p(X\geq t)$ are the same.

Sec 1.9. Special Exp.

Mean: $\mathbb{E}(X) = \mu$

Variance:

$$V(X) = \mathbb{E}((X-\mu)^2)$$
$$= E(X^2 - 2\mu X + \mu^2)$$
$$= E(X^2) - \mu^2$$
$$= E(X^2) - [E(X)]^2$$

1) $E(X)$, $V(X)$ may be non-finite.

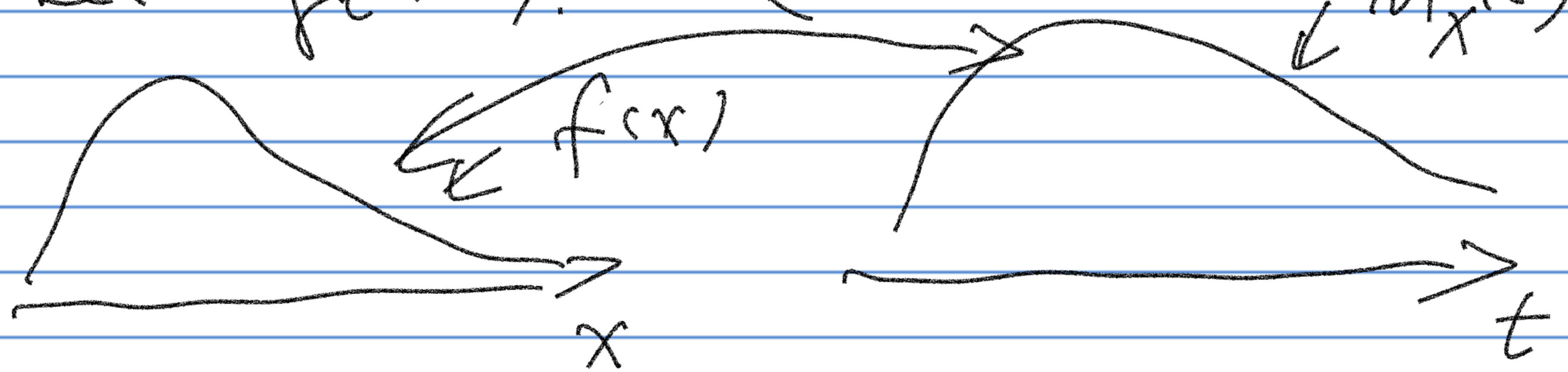
2) $V(X) \geq 0 \Rightarrow E(X^2) \geq [E(X)]^2$

Def of M.G.F.:

$$M_X(t) = E(e^{tX}) \leftarrow g_t(X)$$

t is a fixed value.

Let $g_t(x) = e^{tx}$



Remarks:

1). $M_X(t)$ may not exist.

$M_X(t) = E(e^{tX})$ may be $= +\infty$.

$M_X(t)$ exists for $t \in (-h, h)$

for some $h \Leftarrow E(X^k) < +\infty$

for all $k = 0, 1, \ldots$

moments

2) $M_X^{(k)}(0) = E(X^k)$ moments.

Pf: $k=1$.

$M_X(t) = E(e^{tX})$

$M_X^{(1)}(t) = E(e^{tX} \cdot X)$

$M_X^{(1)}(0) = E(X)$

$M_X^{(2)}(t) = E(e^{tX} \cdot X \cdot X)$

$= E(e^{tX} \cdot X^2)$

$$M^{(2)}(0) = \mathbb{E}(X^2)$$

$$\vdots$$

$$M^{(k)}(t) = \mathbb{E}(e^{tX} \cdot X^k)$$

$$M^{(k)}(0) = \mathbb{E}(X^k)$$

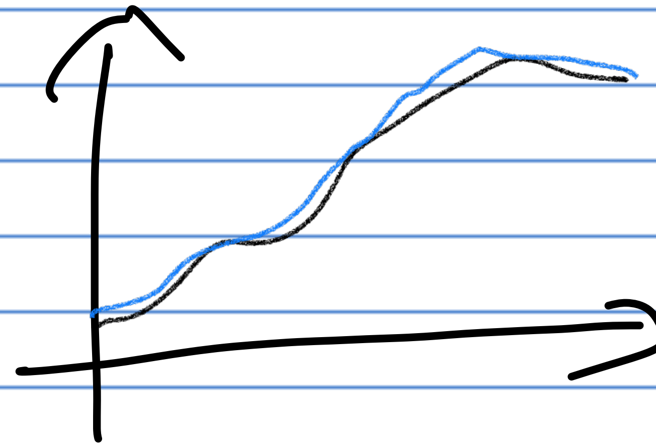
Thm: (Thm 1.9.1.)

Let X & Y be two r.v.

$F_X(x) = F_Y(y)$ for all x & y

$\iff M_X(t) = M_Y(t)$ for $t \in (-h, h)$ for some $h > 0$.

$\iff$



Characteristic Function (always exist)

i is the imaginary number. ($i^2 = -1$)

$$C_X(t) = E\left(e^{itx}\right)$$

$$= E\left(\cos(tx) + i\sin(tx)\right)$$

$$= E\left(\cos(tx)\right) + iE\left(\sin(tx)\right)$$

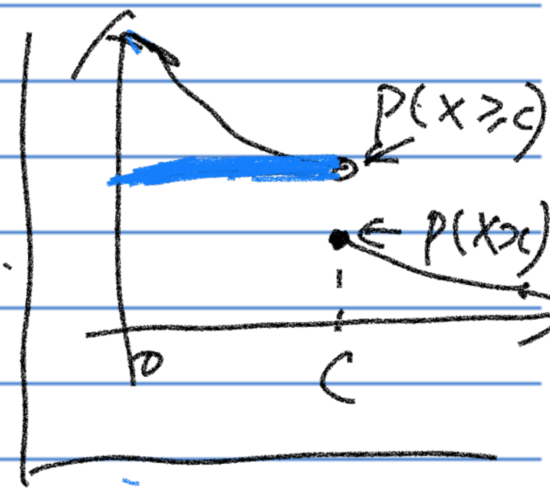
$$< +\infty, \quad |\cos(tx)| < 1, \quad |\sin(tx)| < 1$$

Section 1.10

Important Inequalities

Markov Inequality

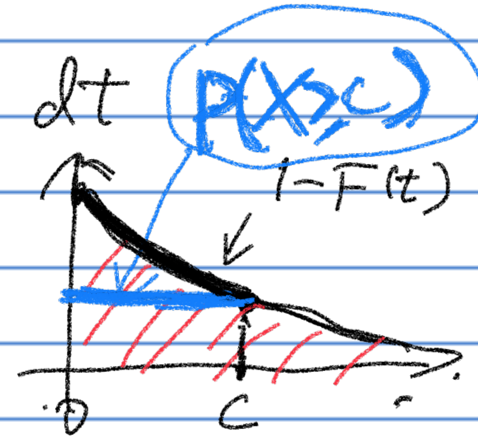
Thm: $X \geq 0$, $P(X \geq c) \leq \dfrac{E(X)}{c}$ for any $c > 0$.



Pf: $E(X) = \int_0^{+\infty} P(X \geq t)\, dt$

$\geq \int_0^c P(X \geq t)\, dt$

$\geq P(X \geq c) \cdot c$



$P(X \geq t) \geq P(X \geq c)$, for $t < c$

$$P(X \geq c) \leq \frac{E(X)}{c}$$

Chebyshev's Inequality:

suppose $V(X)$ exists, $\mu = E(X)$

$$P(|X-\mu| \geq c) \leq \frac{V(X)}{c^2}$$

pf: LHS

$$= P(|X-\mu|^2 \geq \quad)$$

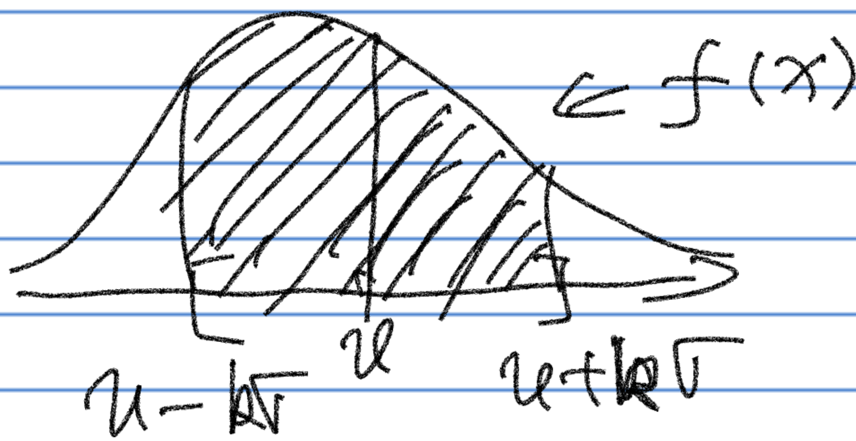
$$\leq \frac{E(|X-\mu|^2)}{c^2}, \text{ by Markov's Ineq.}$$

$$= \frac{V(X)}{c^2}$$

Another form of Chebyshev's Ineq.

$$c = k \cdot \sigma = k \cdot \sqrt{V(X)}$$

$$P(|X - \mu| > k \cdot \sigma) \le \frac{V(X)}{k^2 \cdot \sigma^2} = \frac{1}{k^2}$$



$k = 3, \quad \frac{1}{k^2} = \frac{1}{9}$

$k = 4, \quad \frac{1}{k^2} = \frac{1}{16}$

Sometimes useful in practice.

$\sigma = sd(X)$

Thm:

If $E(X^m) < +\infty$ exists

then $E(X^k) < +\infty$ for all $k \le m$.

pf:

$$E(|X|^k) = E(\underbrace{|X|^k 1(|X| \le 1)}_{\le 1}) + E(|X|^k 1(|X| > 1)) \le E(1(|X| \le 1)) + E(|X|^m 1(|X| > 1))$$

$$\le P(|X| \le 1) + E(|X|^m) < \infty$$

$E(X^k)$ is finite

exist. finite. $< +\infty$. §.1.2.1.

non-existent, non-finite.

$$\begin{cases} +\infty \\ \text{undefined: } +\infty - +\infty \end{cases}$$