Lecture 9

Longhai Li, October 7, 2021

plans:

1) Jensen Inequality Sec 1.10

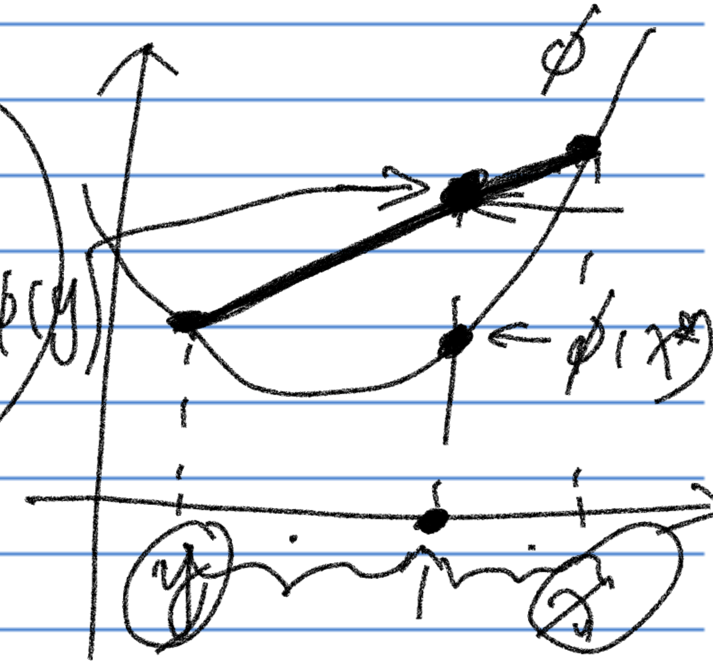
2) Joint distribution Sec 2.1

Jensen Inequality.

Convex functions

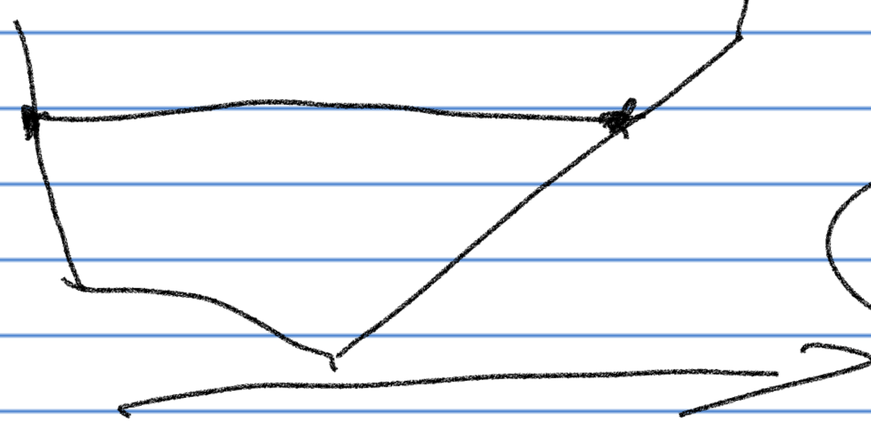
$$\phi(rx+(1-r)y) \le r\,\phi(x)+(1-r)\,\phi(y)$$

$r\phi(x)+(1-r)\phi(y)$



$x^* = rx+(1-r)y$

$r = \frac{1}{2}$ $\quad r \in (0,1)$



$\phi'(x)$ exists

ϕ is convex iff

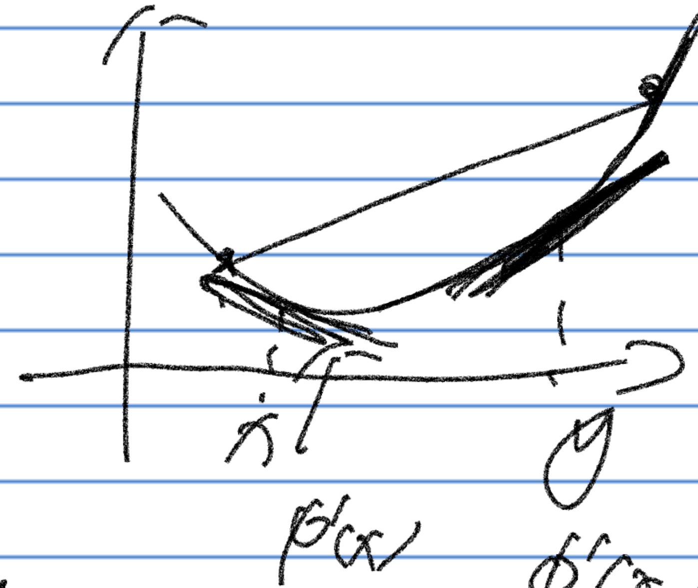
$\phi'(x)$ is non-decreasing

$\phi'(x) \le \phi'(y)$ if $x \le y$

$\phi''(x)$ exists.

ϕ is convex iff

$\phi''(x) \geq 0$



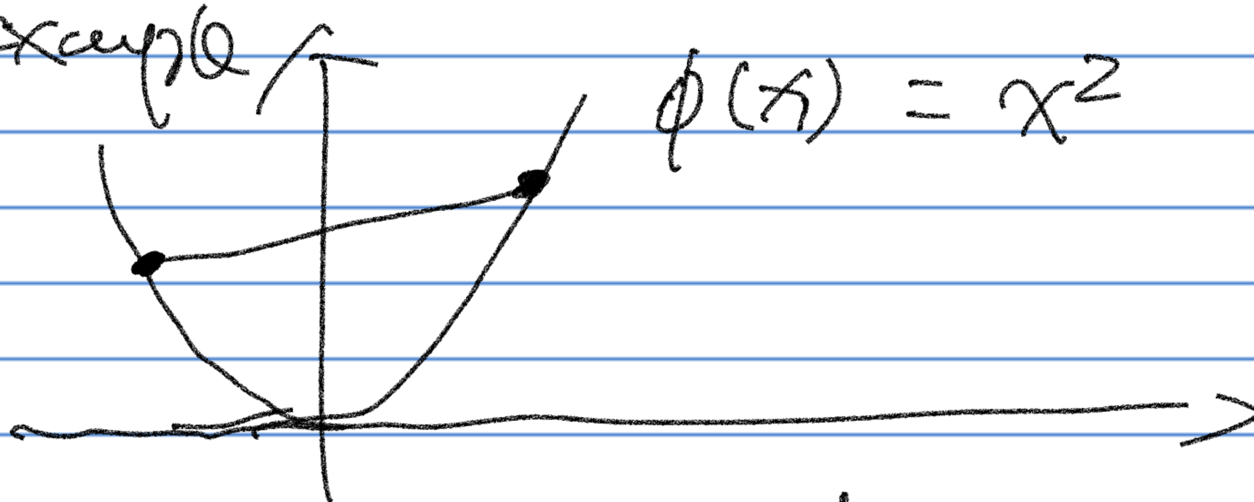
Example:

$\phi(x) = -\log(x)$

$\phi'(x) = -\frac{1}{x}$ increasing

$\phi''(x) = \frac{1}{x^2} \geq 0$ positive.

Example



$\phi(x) = x^2$

$\phi'(x) = 2x$, increasing

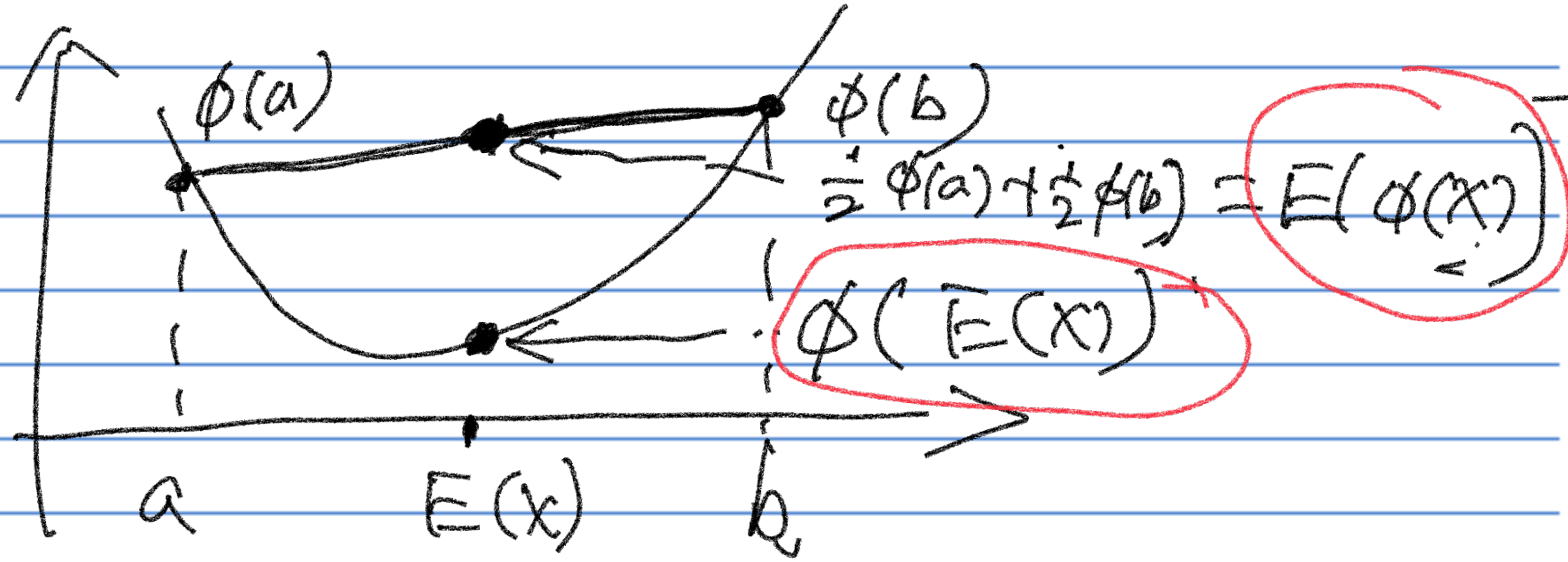
$\phi''(x) = 2 , > 0$

Jensen Inequality

ϕ is a convex function.

$$E(\phi(X)) \geq \phi(E(X))$$

X	a	b
P	$\frac{1}{2}$	$\frac{1}{2}$

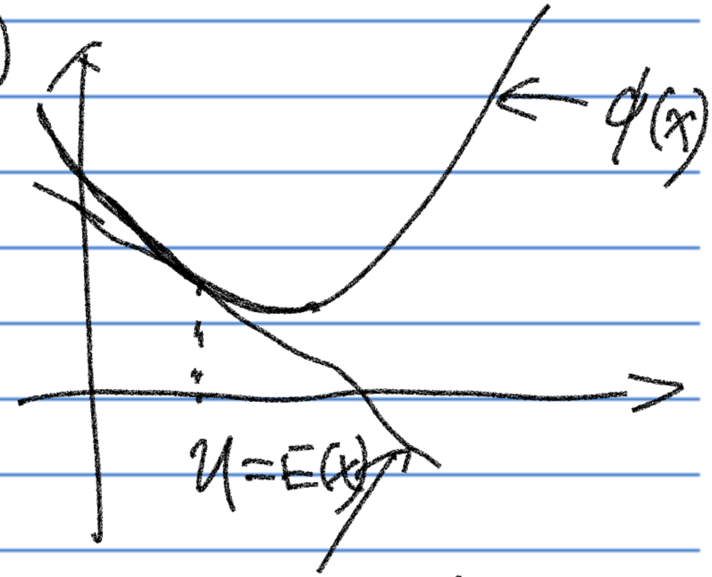


Pf: Assuming $\phi'(u)$ exists, $u = E(X)$

$\phi(x) \geq \phi(u) + \phi'(u)(x-u)$ for all x

$\phi(X) \geq \phi(u) + \phi'(u)(X-u)$ for all X

$$E(\phi(X)) \geq \phi(u) + \phi'(u)\,E(X-u)$$
$$= \phi(u) + \phi'(u)\underbrace{(E(X)-u)}_{0}$$
$$= \phi(u)$$
$$= \phi(E(X)).$$

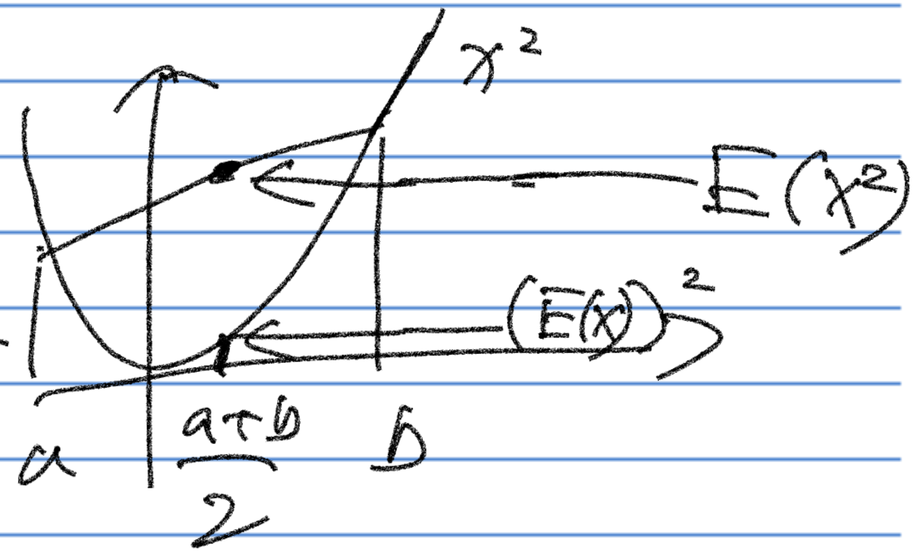


$\ell(x) = \phi(u) + \phi'(u)(x-u)$

Examples:

1) $\phi(X) = X^2$

$$E(X^2) \geq [E(X)]^2$$

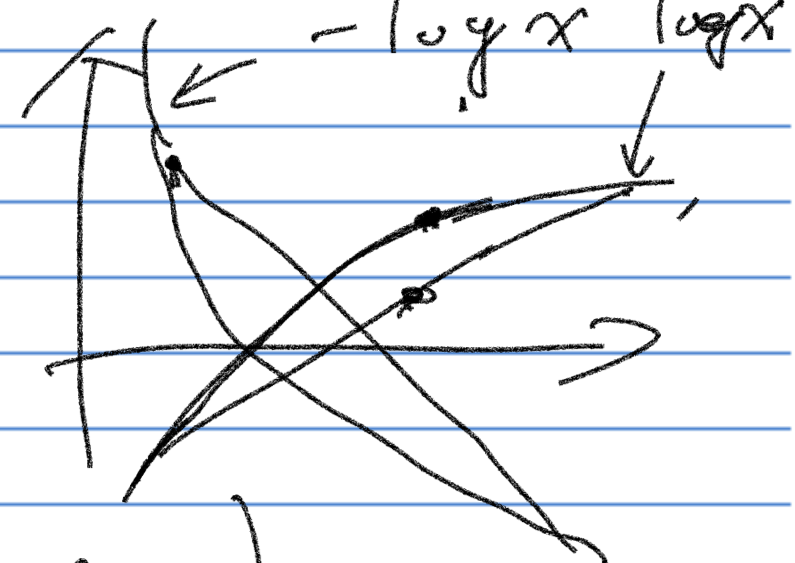


$$E(X^2) \geq (E(X))^2$$

$$V(X) = E(X^2) - (E(X))^2 \geq 0$$

2)

$$E(-\log(X)) \geq -\log(E(X))$$



$$E(\log(X)) \leq \log(E(X))$$

suppose $X \sim$

x	a_1	a_2	$\cdots$	a_n
p	$\frac{1}{n}$	$\frac{1}{n}$		$\frac{1}{n}$

$a_i > 0$

$$E(\log(X)) \le \log(E(X))$$

$$\log(X) \sim \begin{array}{c|ccc} & \log a_1 & \cdots\cdots & \log a_n \\ \hline & \frac{1}{n} & & \frac{1}{n} \end{array}$$

$$E[\log(X)] = \frac{\sum_{i=1}^{n} \log a_i}{n}$$

$\log x + \log y$

$= \log(x \cdot y)$

$$= \frac{\log\left(\prod_{i=1}^{n} a_i\right)}{n} = \log\left(\underline{\left[\prod a_i\right]^{\frac{1}{n}}}\right)$$

$$\log(E(X))$$

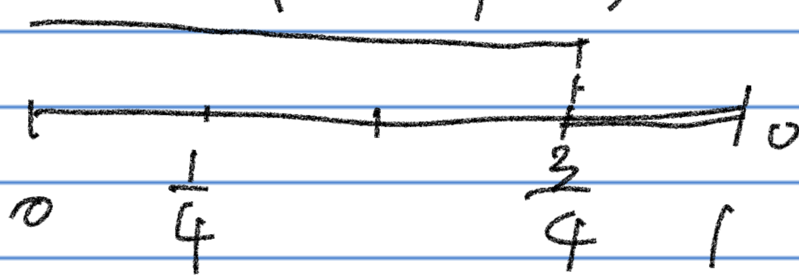
$$= \log\left(\frac{a_1 + \cdots + a_n}{n}\right)$$

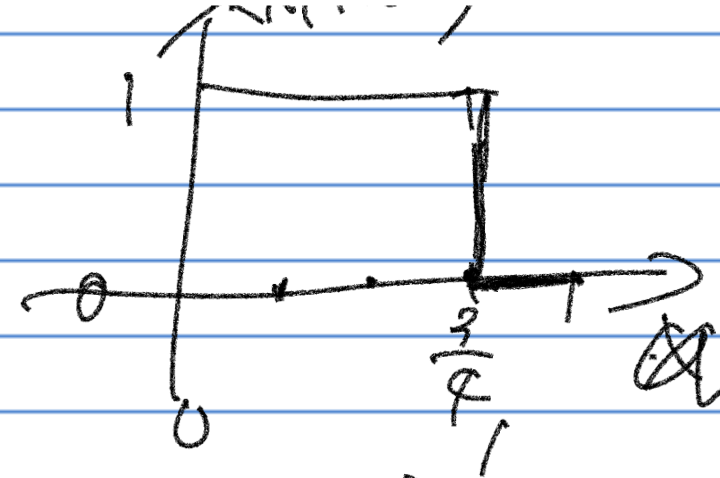
$$\log\left(\left[\prod_{i=1}^{n} a_i\right]^{\frac{1}{n}}\right) \leq \log\left(\frac{\sum a_i}{n}\right)$$

$$\left[\prod_{i=1}^{n} a_i\right]^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} a_i}{n}.$$

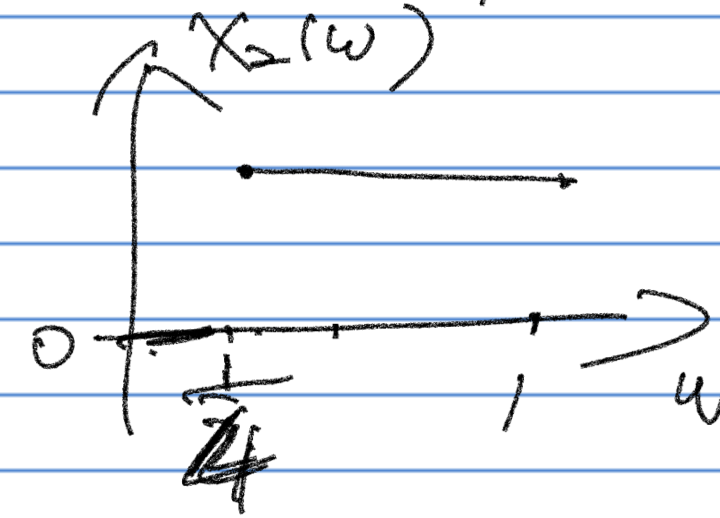
$$G.M. \leq A.M.$$

$\omega \sim \text{Unif}([0,1])$





$X_1(\omega) = \mathbb{1}\left(\omega \le \frac{3}{4}\right)$



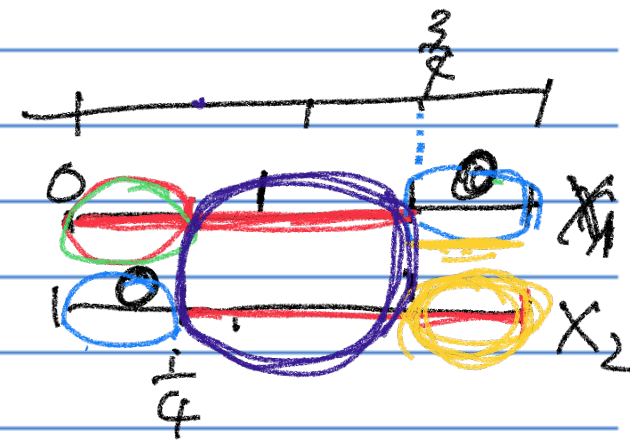
$X_2(\omega) = \mathbb{1}\left(\omega \ge \frac{1}{4}\right)$

Joint P.M.F.

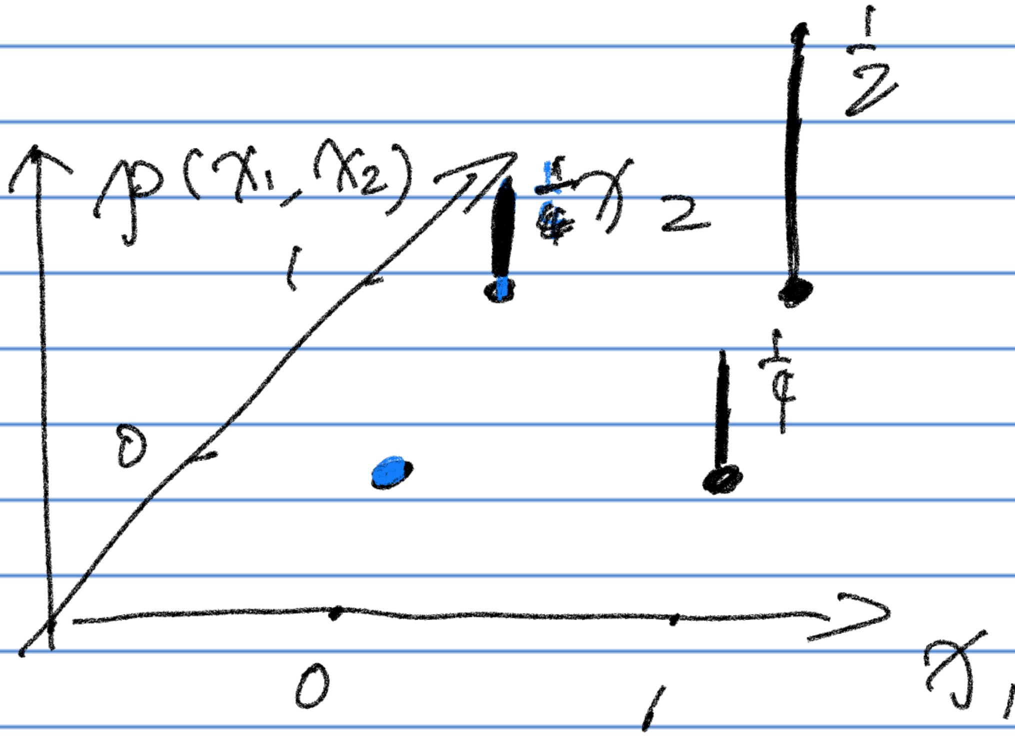
$$p(x_1, x_2) = P(X_1 = x_1,\ X_2 = x_2)$$

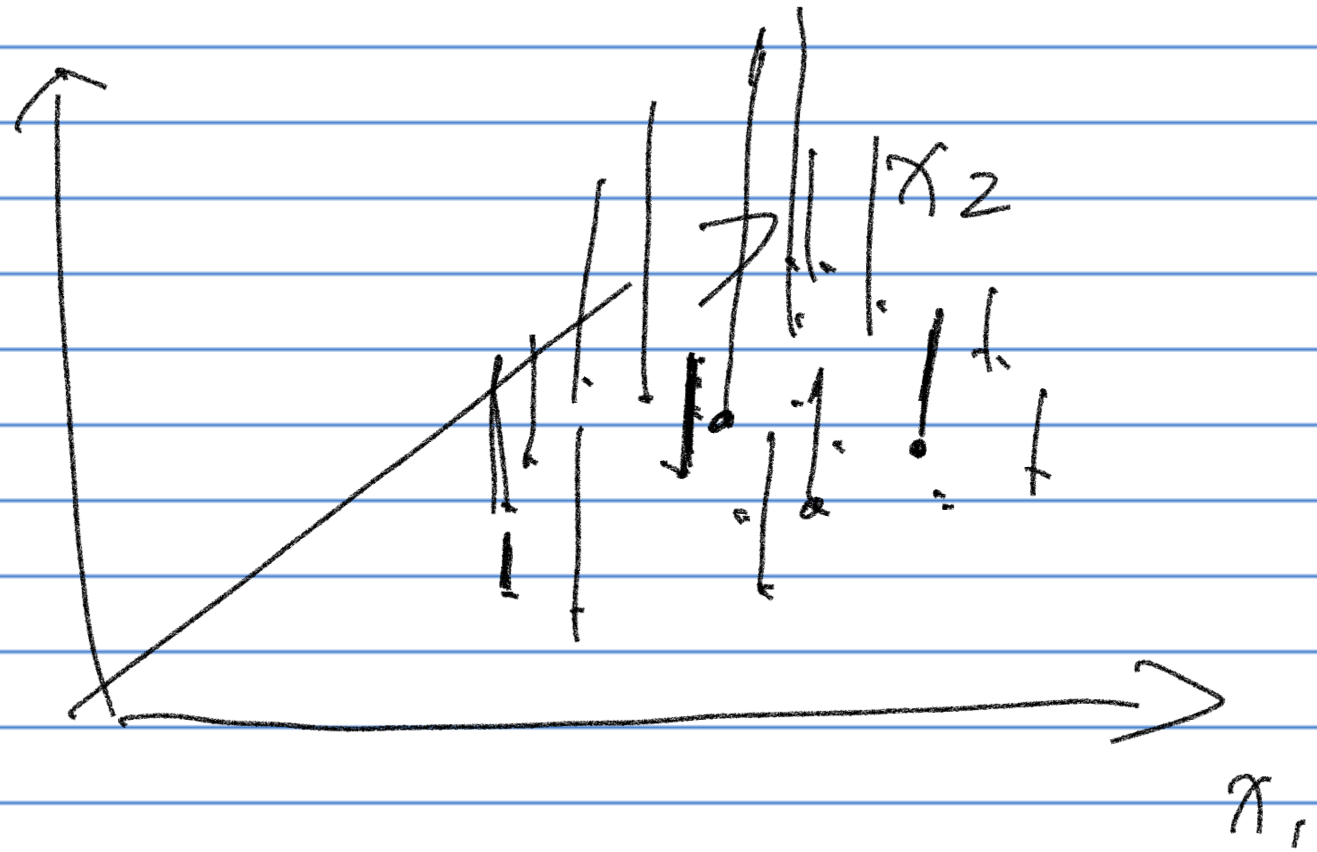
for all possible values that X_1 & X_2 can take.

$x_2 \backslash x_1$	0	1
0	0	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{2}$



NOT. Indept

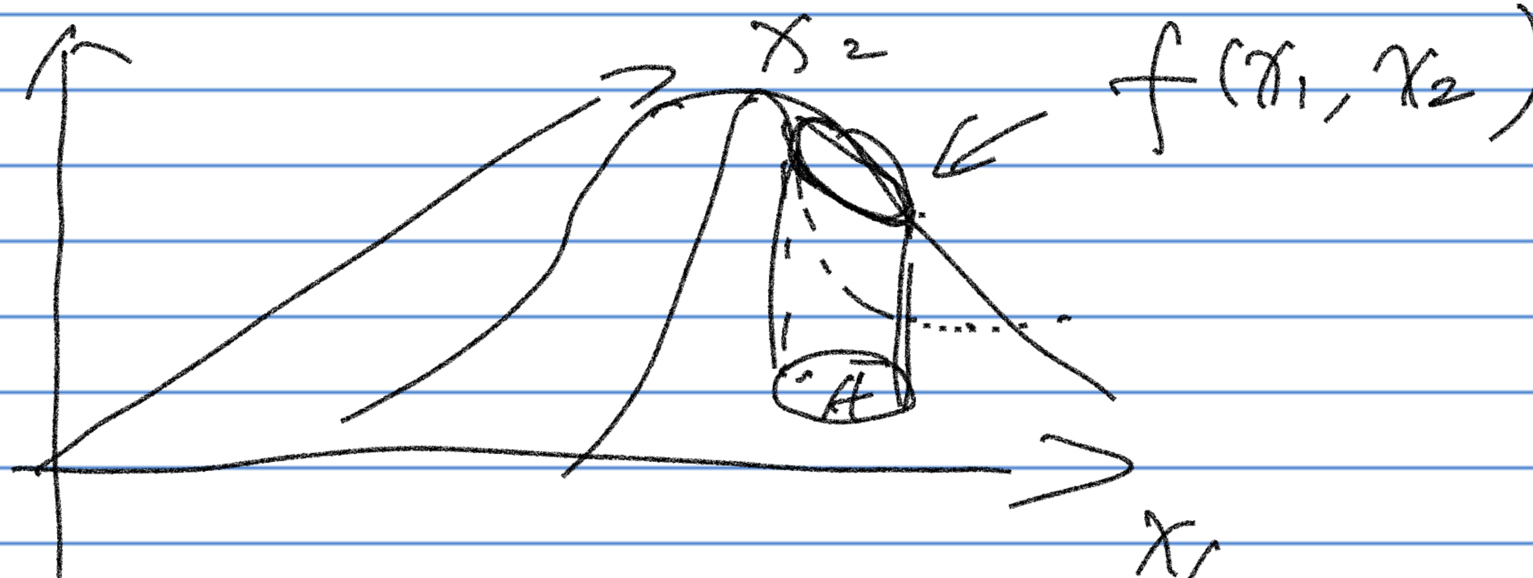

$p(x_1, x_2)$
x_2
x_1
0
1
0
1
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{1}{4}$



x2
x1

Joint P.D.F.

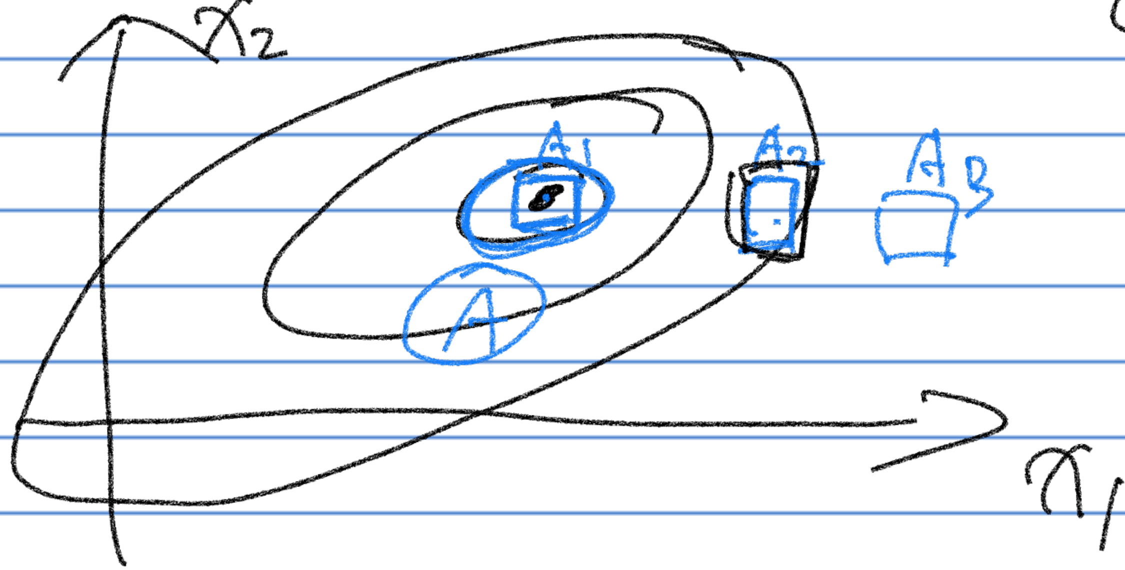
X_1, X_2 are both continuous



$P(X_1 = x_1, X_2 = x_2) = 0$

We say $f(x_1, x_2)$ is a joint P.D.F. of (X_1, X_2) if

$$P((X_1, X_2) \in A) = \iint_A f(x_1, x_2)\,dx_1\,dx_2$$



Contour of $f(x_1, x_2)$

Marginal P.M.F. & P.D.F.

Example

$x_2 \backslash x_1$	0	1	$p_{X_2}(x_2)$
0	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$p_{X_1}(x_1)$	$\frac{1}{4}$	$\frac{3}{4}$	

$$p_{X_2}(x_2) = \sum_{x_1} p(x_1, x_2)$$

$$p_{X_1}(0) = p(0,0) + p(0,1)$$

$$= \frac{1}{4}$$

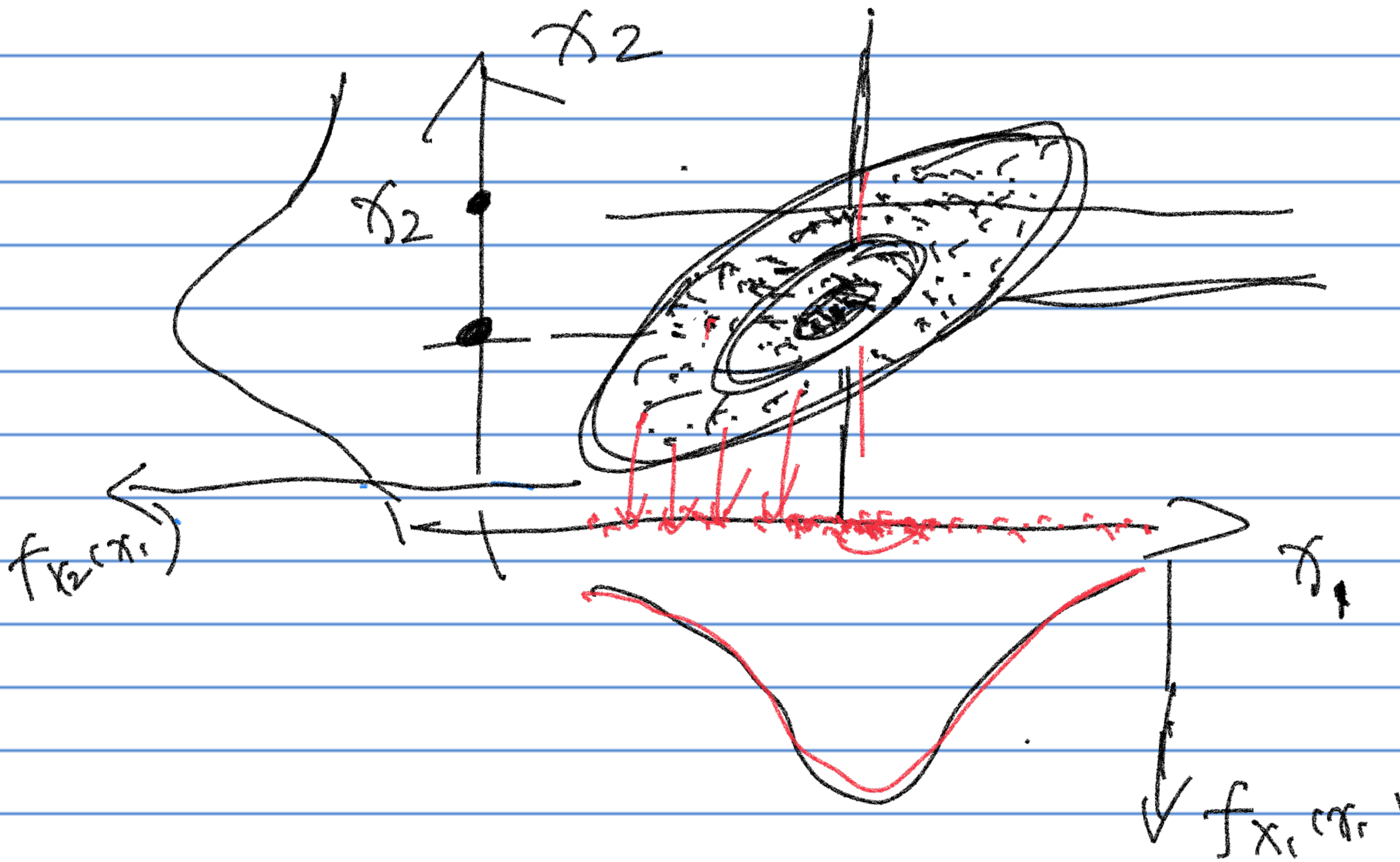
$$p_{X_1}(x_1) = \sum_{x_2} p(x_1, x_2)$$

Marginal P.D.F.

Suppose $f(x_1, x_2)$ is a joint P.D.F. of (X_1, X_2).

$$f_{X_1}(x_1) = \int_{-\infty}^{+\infty} f(x_1, x_2)\, dx_2$$

$$f_{X_2}(x_2) = \int_{-\infty}^{+\infty} f(x_1, x_2)\, dx_1.$$



x_2
x_2
x_1
$f_{X_2}(x_1)$
$f_{X_1}(x_1)$