Lecture 10 and 11

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plans:

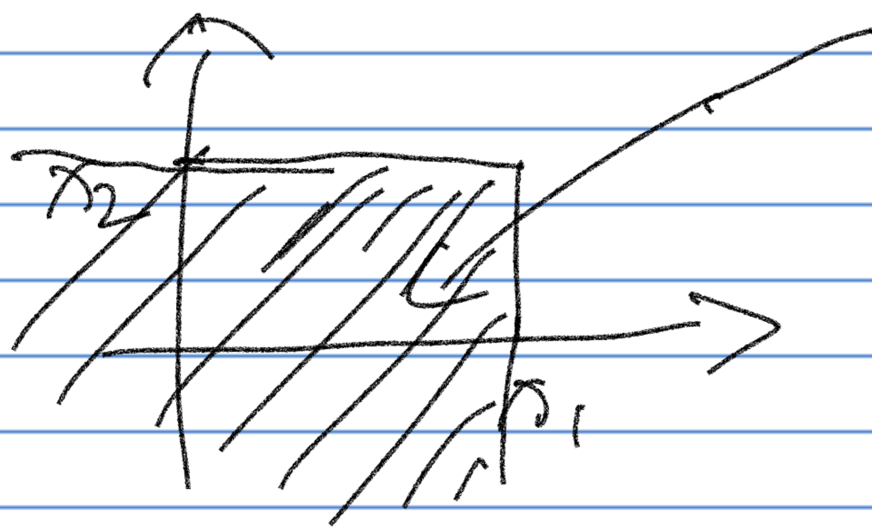
1. Sec 2.1.1. expectation.

2. Sec 2.1.2. transformation.

Joint C.D.F.

$$F(x_1, x_2) = P(X_1 \le x_1,\ X_2 \le x_2)$$

$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(t_1, t_2)\, dt_1\, dt_2$$



Expectation:

Def: (X_1, X_2) has a p.m.f. $p(x_1, x_2)$

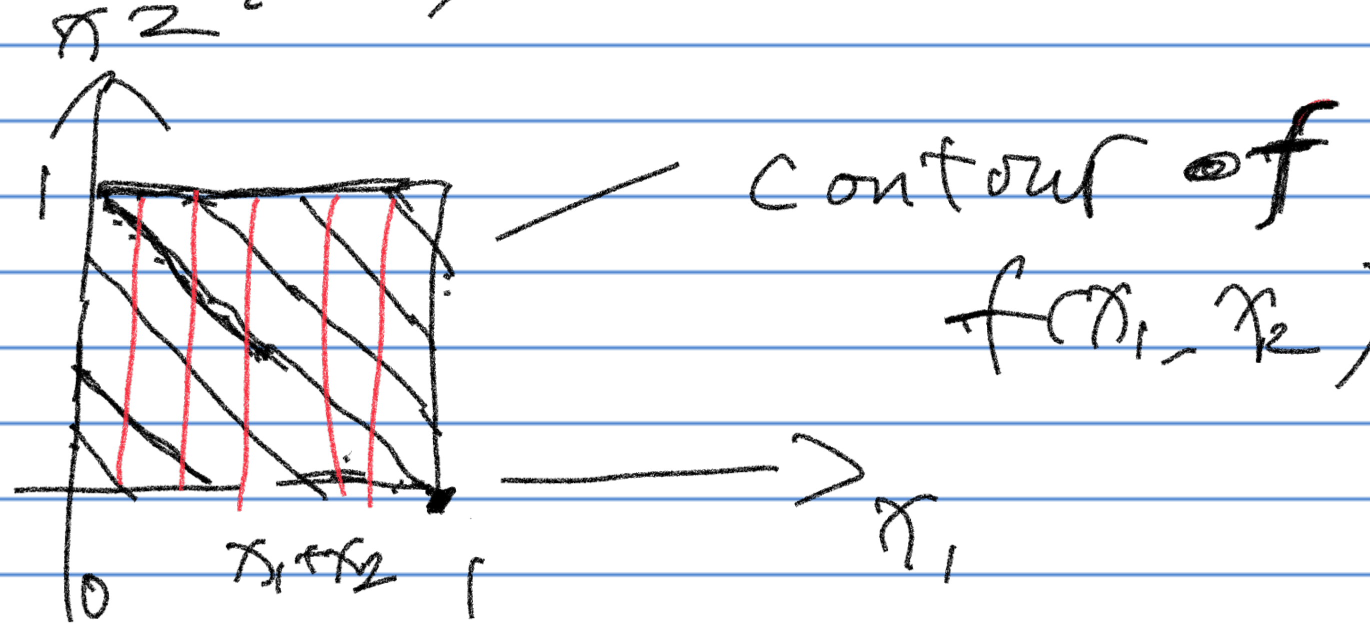
$$E(g(X_1, X_2)) = \sum_{x_1, x_2} g(x_1, x_2)\, p(x_1, x_2)$$

Def: (X_1, X_2) has a p.d.f. $f(x_1, x_2)$

$$E(g(X_1, X_2)) = \iint g(x_1, x_2)\, f(x_1, x_2)\, dx_1\, dx_2$$

Example:

$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & \text{if } (x_1, x_2) \in [0,1] \times [0,1] \\ 0, & \text{o.w.} \end{cases}$$



$$g_1(X_1, X_2) = X_1 \qquad E(g_1(X_1, X_2))$$

$$E(g_1(X_1, X_2)) = E(X_1)$$

$$= \int_0^1 \int_0^1 x_1 \, f(x_1, x_2) \, dx_1 \, dx_2$$

$$= \int_0^1 \int_0^1 x_1 \cdot (x_1 + x_2) \, dx_1 \, dx_2 .$$

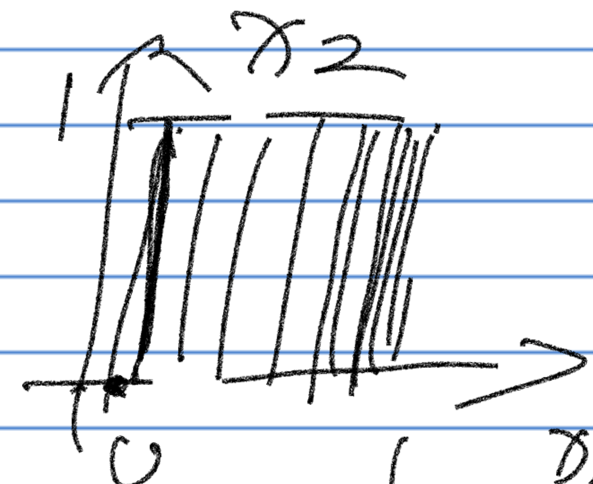
$$= \int_0^1 \int_0^1 x_1^2 \, dx_1 \, dx_2 + \underline{\int_0^1 \int_0^1 x_1 x_2 \, dx_1 \, dx_2}$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

where

$$\int_0^1 \left[\int_0^1 x_1^2 \, dx_2 \right] dx_1$$

$$= \int_0^1 x_1^2 \times 1 \, dx_1 = \frac{x_1^3}{3} \Big|_0^1 = \frac{1}{3}$$



$$\int_0^1 \left[\int_0^1 x_1 \, x_2 \, dx_1\right] dx_2$$

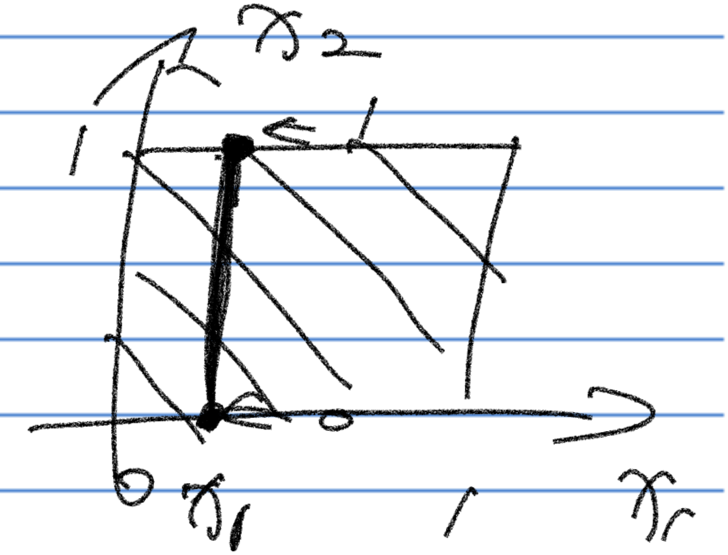
$$= \int_0^1 \left[x_2 \int_0^1 x_1 \, dx_1\right] dx_2$$

$$= \int_0^1 \left[x_2 \cdot \left(\underbrace{\frac{x_1^2}{2}}\right|_0^1\right] dx_2$$

$$= \int_0^1 \frac{1}{2} x_2 \, dx_2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$E(X_1) = ?$ To find $f_{X_1}(x_1)$ first.

For $x_1 \in [0,1]$



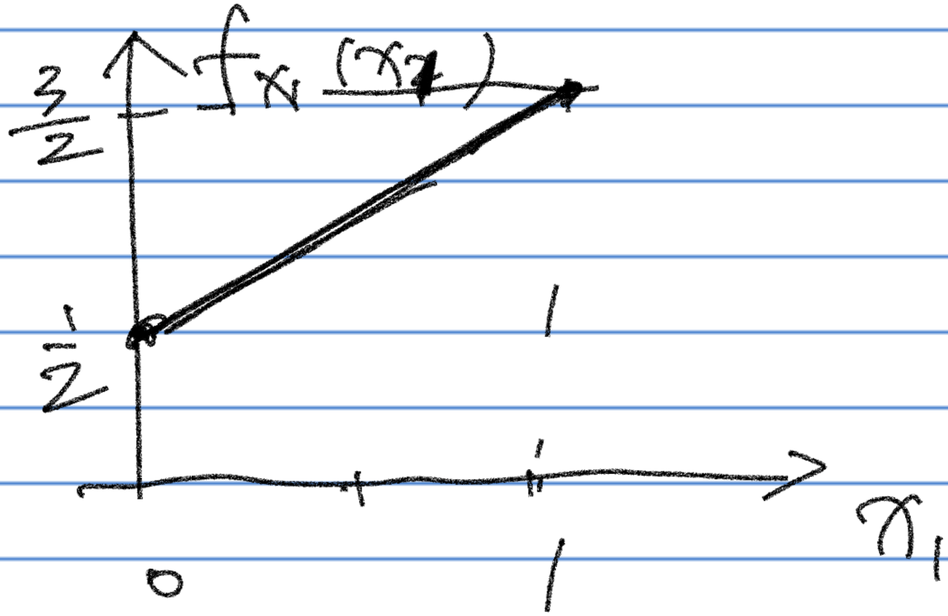
$f_{X_1}(x_1)$

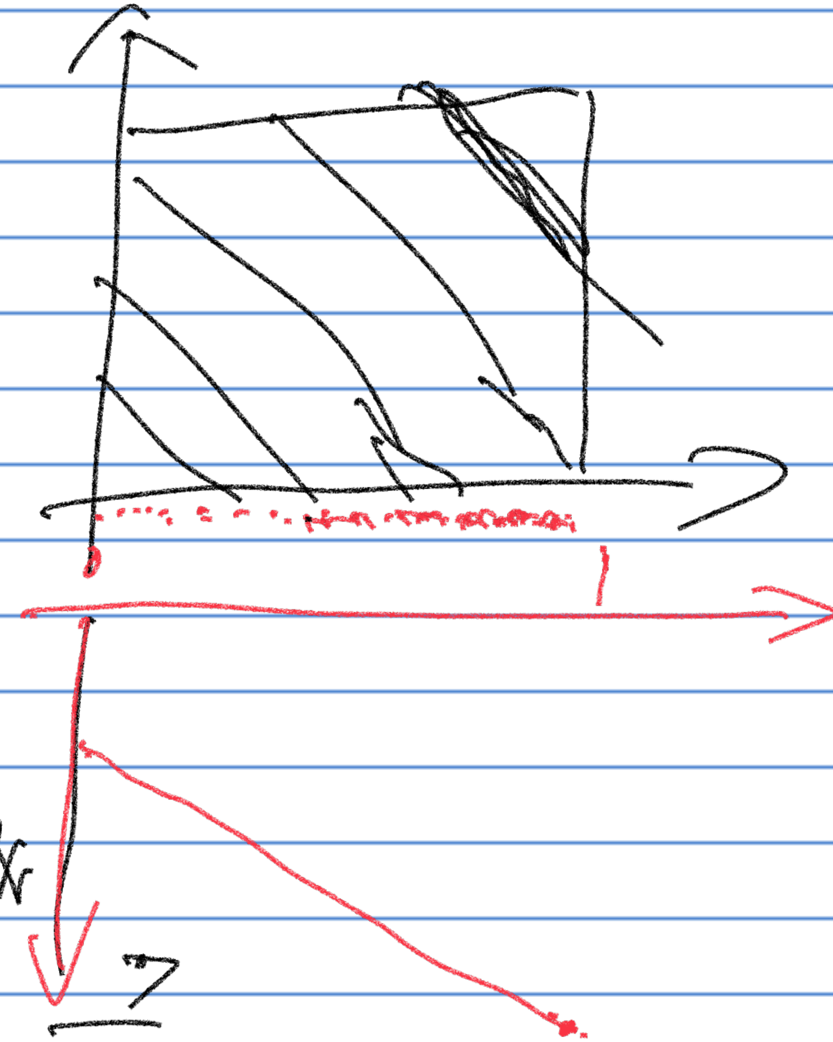
$$= \int_{-\infty}^{+\infty} f(x_1, x_2)\, dx_2$$

$$= \int_0^1 (x_1 + x_2)\, dx_2$$

$$= \int_0^1 x_1\, dx_2 + \int_0^1 x_2\, dx_2$$

$$= x_1 \cdot 1 + \frac{1}{2} = x_1 + \frac{1}{2}$$





$$f_{X_1}(x_1) = x_1 + \frac{1}{2}$$

$$E(X_1) = \int_0^1 x_1 \cdot \left(x_1 + \frac{1}{2}\right) dx_1$$

$$= \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{12}$$

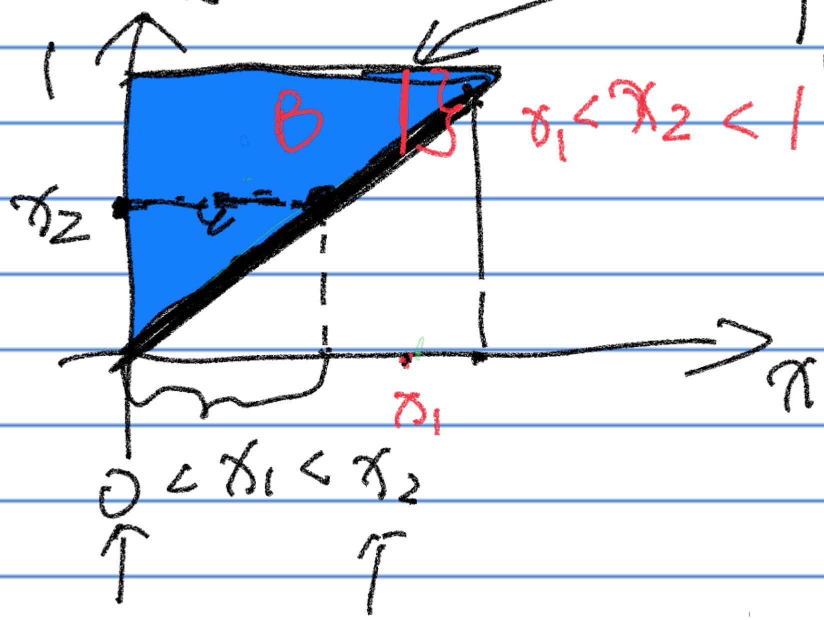
$$\int\int x_1 \, f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int \left[\int x_1 \, f(x_1, x_2)\, dx_2 \right] dx_1$$

$$= \int x_1 \cdot \underbrace{\left[\int f(x_1, x_2)\, dx_2 \right]}_{f_{X_1}(x_1)} dx_1$$

Example:

$$f_{X_1X_2}(x_1, x_2) = \begin{cases} 8x_1x_2 & 0 < x_1 < x_2 < 1 \end{cases}$$



$$E(X_1^0 X_2^2) = \iint_B x_1 x_2^2 \cdot 8x_1 \cdot x_2 \, dx_1 \, dx_2$$

$$= \int_0^1 \int_{x_1}^1 \left[8x_1^2 x_2^3 \, dx_2\right] dx_1$$

$$= \int_0^1 8x_1^2 \cdot \left[\int_{x_1}^1 x_2^3 \, dx_2\right] dx_1$$

$$= \int_0^1 8x_1^2 \tfrac{1}{4}(1 - x_1^4) \, dx_1$$

$$= \int_0^1 2x_1^2 \, dx_1 - \int_0^1 2x_1^6 \, dx_1$$

$$= \frac{2}{3} - 2 \times \frac{1}{7} = \frac{8}{21}$$

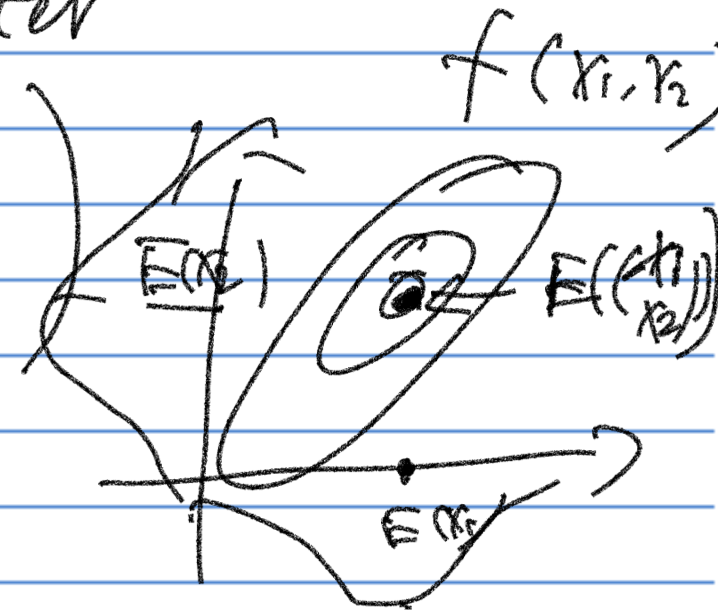
M.G.F. of Random Vector

Def.:

$$M_{(X_1, X_2)}(t_1, t_2) = E\left(e^{t_1 X_1 + t_2 X_2}\right)$$

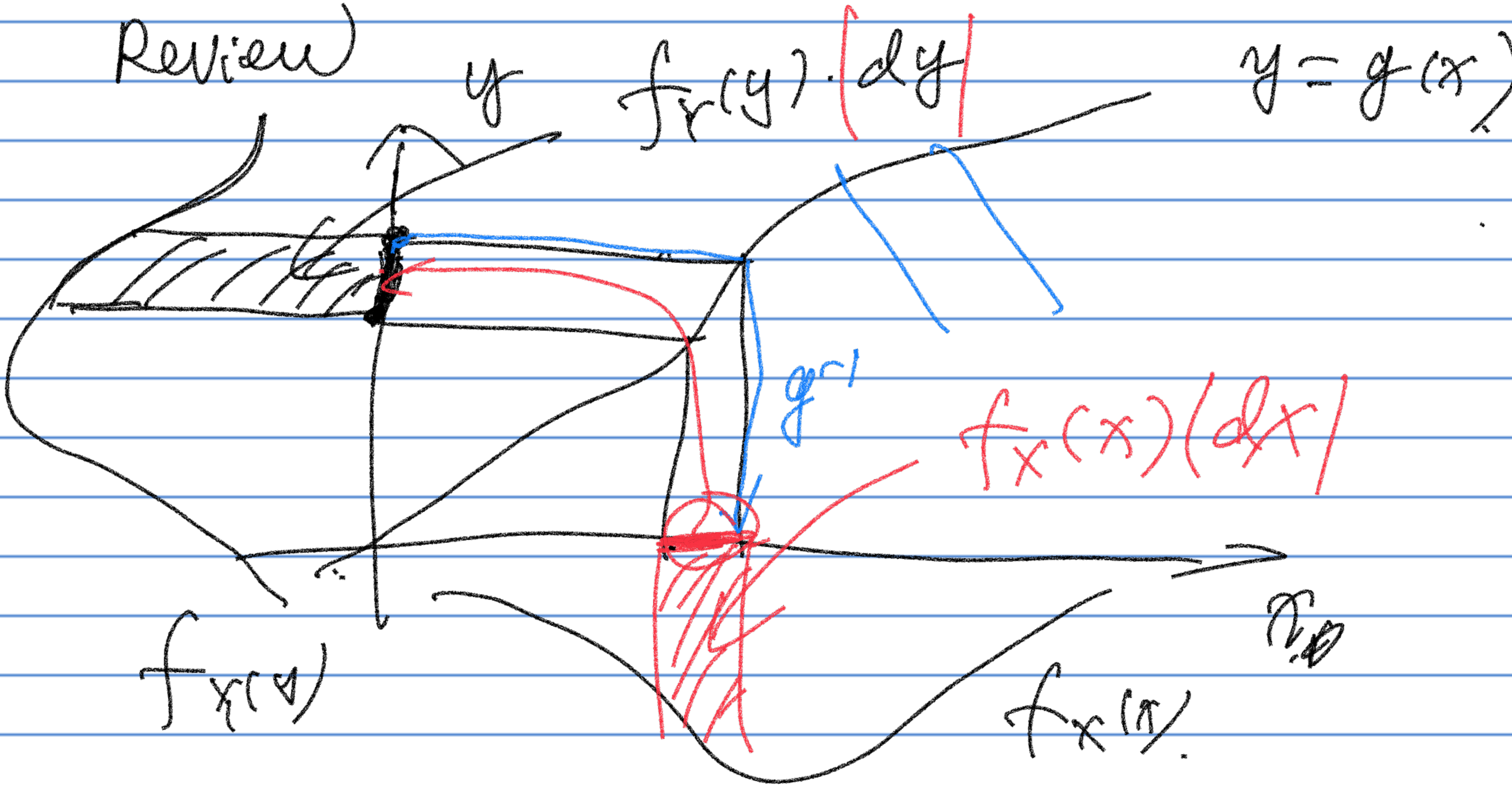
Expectation of Random Vector

$$E\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\right) = \begin{pmatrix} E(X_1) \\ E(X_2) \end{pmatrix}$$



Transformation of random vector

Review

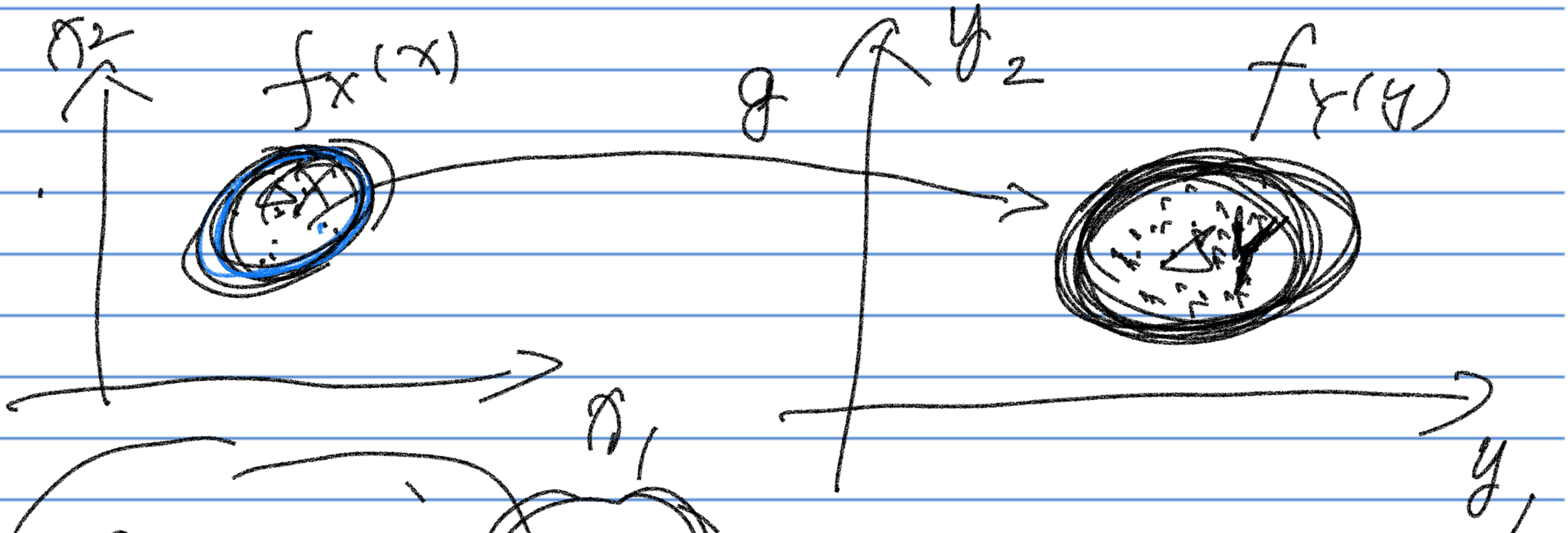


- $f_Y(y)\,|dy| = f_X(x)\,|dx|$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

$$= f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$$

Jacobian.



$$f_{X_1,X_2}(x_1,x_2)\,|\Delta x| = f_{Y_1,Y_2}(y_1,y_2)\,|\Delta y|$$

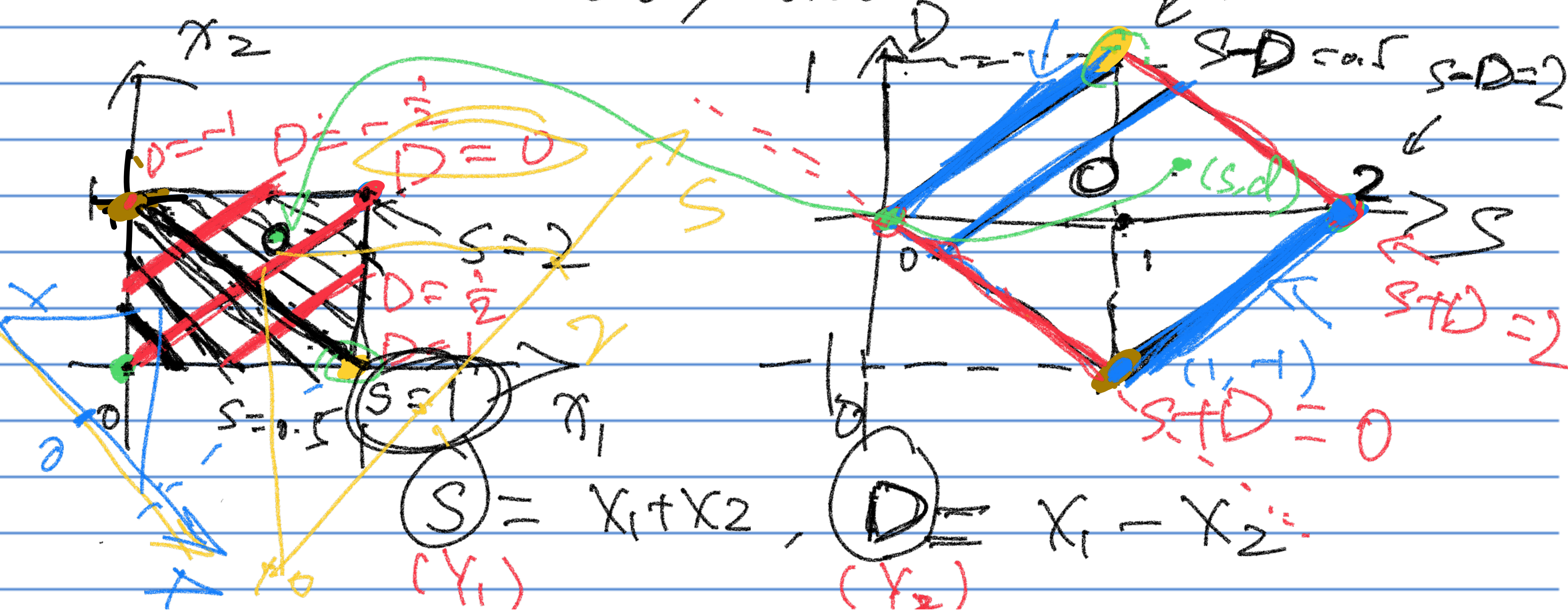
$$f_{(Y_1,Y_2)}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)\,\frac{|\Delta x|}{|\Delta y|}$$

$$\frac{|\partial x|}{|\partial y|} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} \quad \checkmark$$

determinent

Example:

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1, & (x_1,x_2) \in [0,1]\times[0,1] \\ 0, & \text{o.w.} \end{cases}$$



$S = X_1+X_2$, $D = X_1 - X_2$

(Y_1) (Y_2)

$$\begin{cases} D = X_1 - X_2 \\ S = X_1 + X_2. \end{cases} \iff \begin{cases} X_1 = \dfrac{D+S}{2} \checkmark \\ X_2 = \dfrac{S-D}{2} \checkmark \end{cases}$$

$$\begin{cases} 0 \le \dfrac{D+S}{2} \le 1 \\ 0 \le \dfrac{S-D}{2} \le 1 \end{cases} \quad \begin{cases} 0 \le \underline{S+D} \le 2 \\ 0 \le \underline{S-D} \le 2 \end{cases}$$

For $0 \le s+d \le 2$, $0 \le s-d \le 2$

$$f_{S,D}(s,d) = f_{X_1,X_2}(x_1, x_2) \cdot J$$

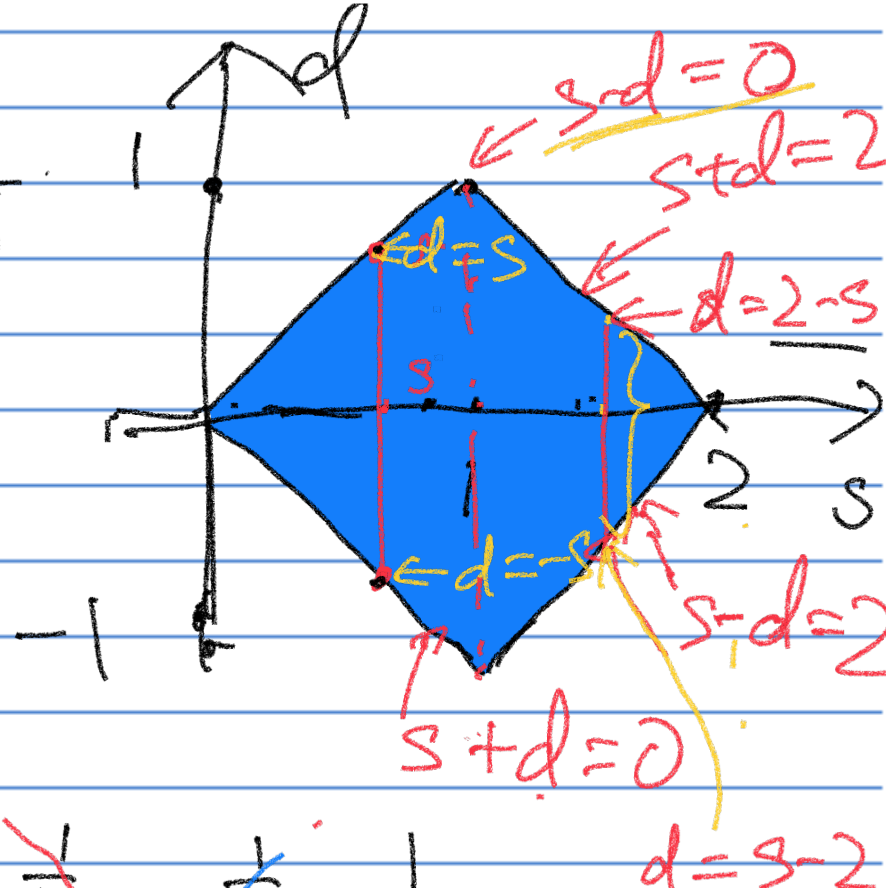
where

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial d} \\ \frac{\partial x_2}{\partial s} & \frac{\partial x_2}{\partial d} \end{vmatrix}^{+}$$

$$x_1 = \frac{s+d}{2}$$

$$x_2 = \frac{s-d}{2}.$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}^{+} = \left| -\frac{1}{4} - \frac{1}{4} \right|^{+} = \frac{1}{2}$$



$$f_{S,D}(s,d) = f_{X_1,X_2}(x_1,x_2) \cdot J$$

$$= 1 \cdot \frac{1}{2}$$

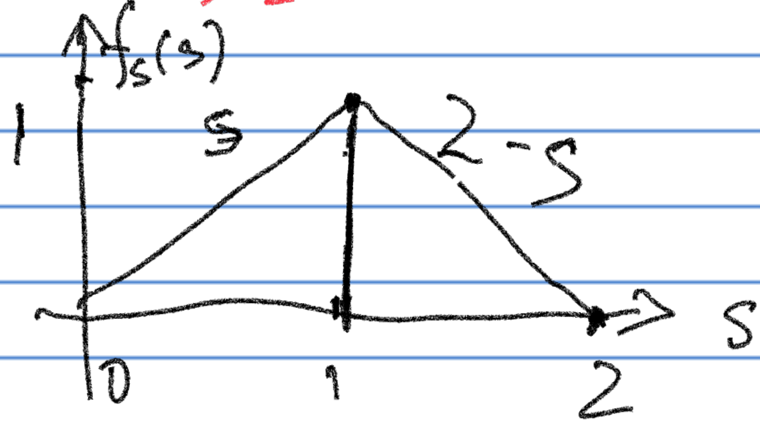
$$= \frac{1}{2}.$$

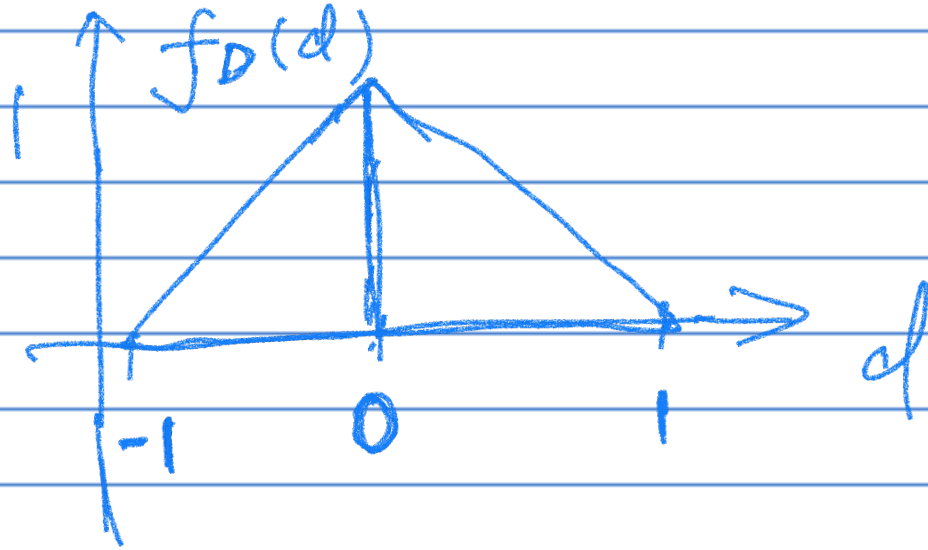
for $0 \le s+d \le 2$, $0 \le s-d \le 2$.

Find the p.D.F. of S & D resp.?

$$f_S(s) = \int_{-\infty}^{+\infty} f_{S,D}(s,d)\, dd$$

$$= \begin{cases} \int_{-s}^{s} \frac{1}{2}\, dd = s, & \text{if } 0 \le s \le 1 \\ \int_{s-2}^{2-s} \frac{1}{2}\, dd = 2-s, & \text{if } 1 \le s \le 2 \end{cases}$$





exercise ?

Ghost method.

Convolution Formula

X_1, X_2 has a joint P.D.F. $f(x_1, x_2)$

What's the P.D.F. of $S = X_1 + X_2$.

$$\begin{cases} S = X_1 + X_2 \\ Y = X_1 \end{cases} \iff \begin{cases} x_1 = y \\ x_2 = s - y \end{cases}$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial y} \\ \frac{\partial x_2}{\partial s} & \frac{\partial x_2}{\partial y} \end{vmatrix}_+ = \cancel{\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}}_+ = 1$$

$$f(s, y) = f_{X_1, X_2}(y, s-y) \cdot 1$$

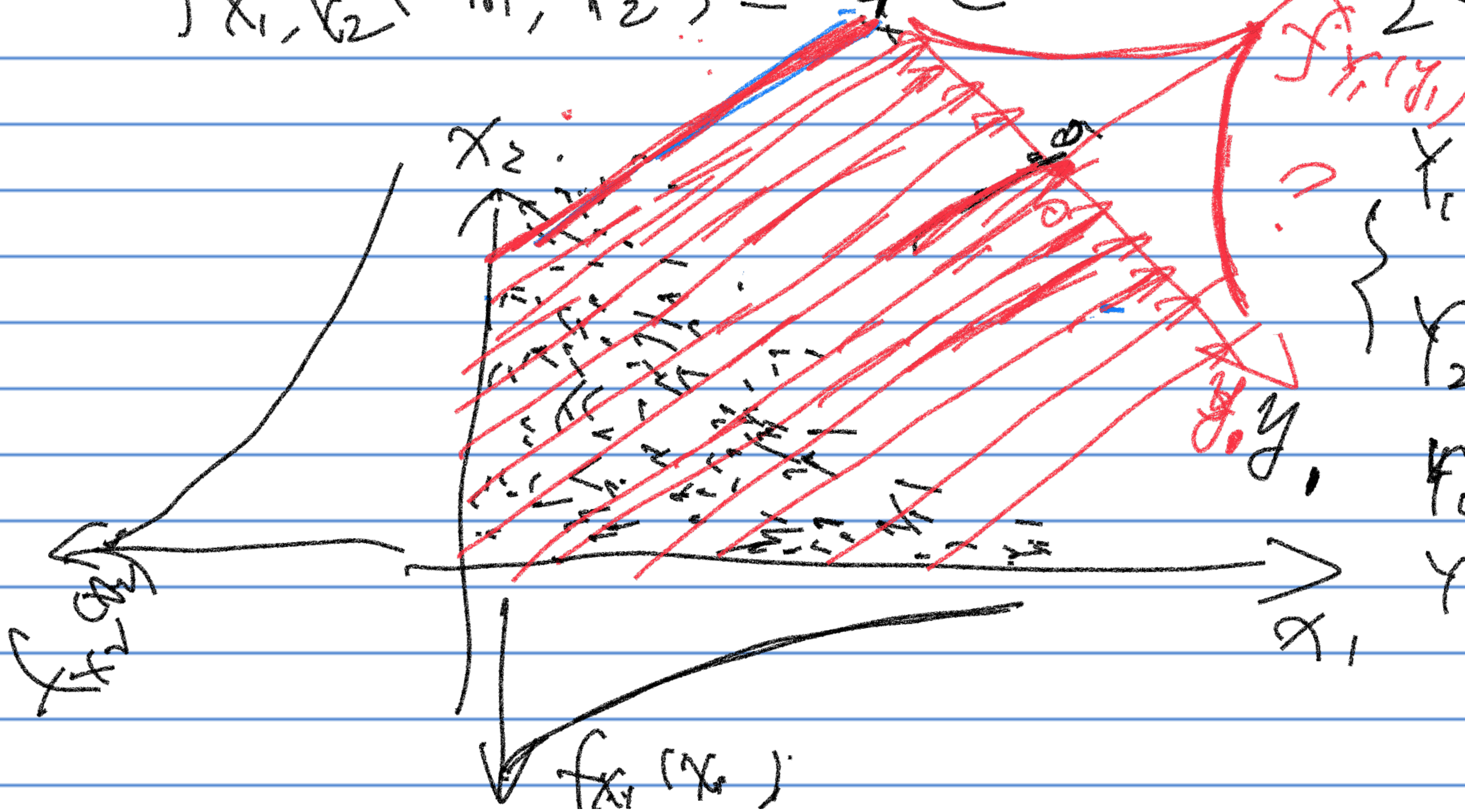
$$f(s) = \int_{-\infty}^{+\infty} f(s, y)\, dy$$

$$= \int_{-\infty}^{+\infty} f_{X_1, X_2}(\underset{X_1}{y}, \underset{X_2}{s-y})\, dy$$

Ghost method.

Example:

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{4} e^{-\frac{x_1+x_2}{2}} = \frac{1}{2} e^{-\frac{x_1}{2}} \cdot \frac{1}{2} e^{-\frac{x_2}{2}}$$



$$\begin{cases} Y_1 = \dfrac{X_1 - X_2}{2} \\ Y_2 = X_2 \end{cases}$$

$Y_1 \in (-\infty, +\infty)$

$Y_2 \in (0, +\infty)$

$$\begin{cases} Y_1 = \dfrac{X_1 - X_2}{2} \\ Y_2 = X_2 \end{cases} \Longleftrightarrow \begin{cases} X_1 = 2Y_1 + Y_2 \\ X_2 = Y_2 \end{cases}$$

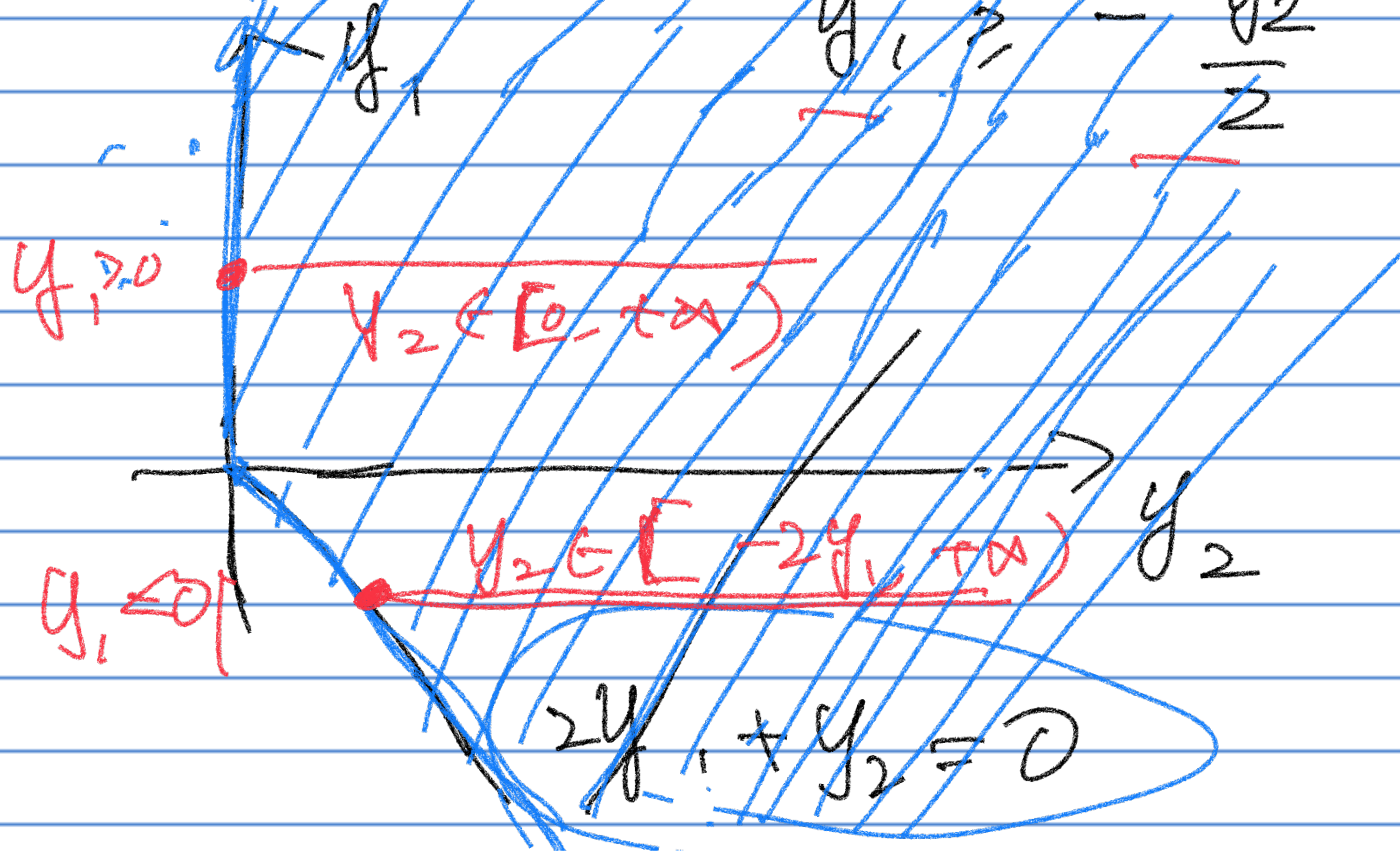
$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

$$2y_1 + y_2 \geq 0, \quad y_2 \geq 0$$

$y_2 \geq 0$

$2y_1 + y_2 \geq 0$

$y_1 \geq -\frac{y_2}{2}$



$$f_Y(y_1, y_2) = f_X(x_1, x_2)\cdot J$$

$$= f_X(\underline{2y_1+y_2}, \underline{y_2})\cdot 2$$

$$= \frac{1}{4} e^{-\frac{1}{2}(2y_1+2y_2)}\cdot 2$$

$$= \frac{1}{2} e^{-(y_1+y_2)}$$

for $y_1 \geq -\frac{y_2}{2}$, $y_2 \geq 0$ $\qquad 2y_1 \geq -y_2$

$$f_{Y_1}(y_1) = \int f_Y(y_1, y_2)\, dy_2 \qquad -2y_1 \leq y_2$$

$$= \begin{cases} \int_0^{+\infty} \frac{1}{2} e^{-y_1}\cdot e^{-y_2}\, dy_2, & y_1 > 0 \\ \int_{-2y_1}^{+\infty} \frac{1}{2} e^{-y_1} e^{-y_2}\, dy_2, & y_1 \leq 0 \end{cases}$$

For $y_1 > 0$

$$f_{Y_1}(y_1) = \int_0^{+\infty} e^{-y_2}\,dy_2 \cdot \frac{1}{2}e^{-y_1}$$

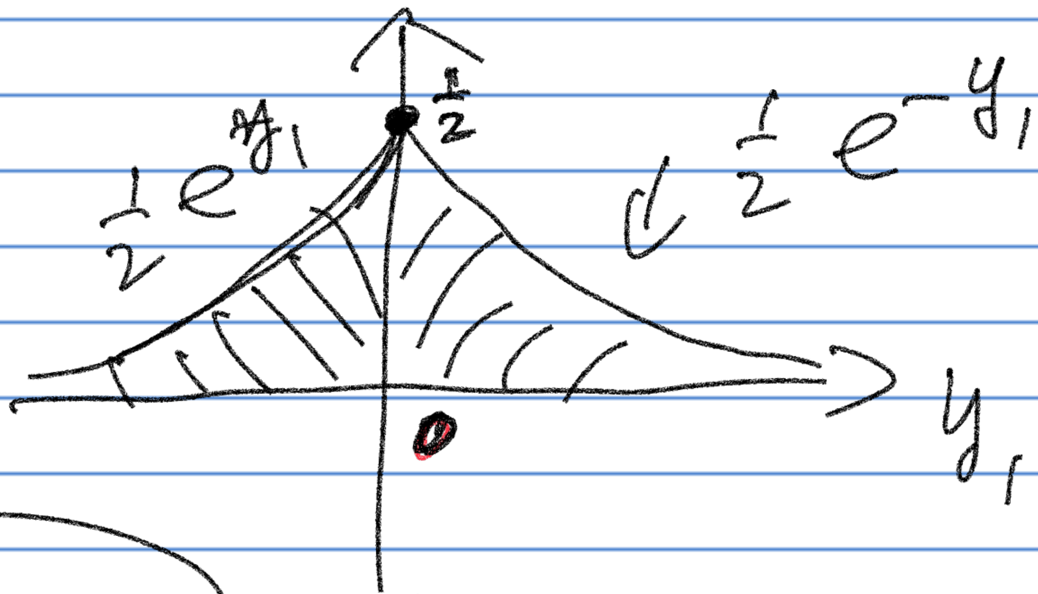
$$= \frac{1}{2}e^{-y_1}$$

For $y_1 \le 0$.

$$f_{Y_2}(y_2) = \int_{-2y_1}^{+\infty} e^{-y_2}\,dy_2 \cdot \frac{1}{2}e^{-y_1}$$

$$= -e^{-y_2}\Big|_{-2y_1}^{+\infty} \cdot \frac{1}{2}e^{-y_1}$$

$$= e^{+2y_1} \cdot \frac{1}{2}e^{-y_1} = \frac{1}{2}e^{y_1}$$



Laplace distribution

LASSO

$$f_{Y_1}(y_1) = \frac{1}{2} e^{-|y_1|}$$