Lecture 12

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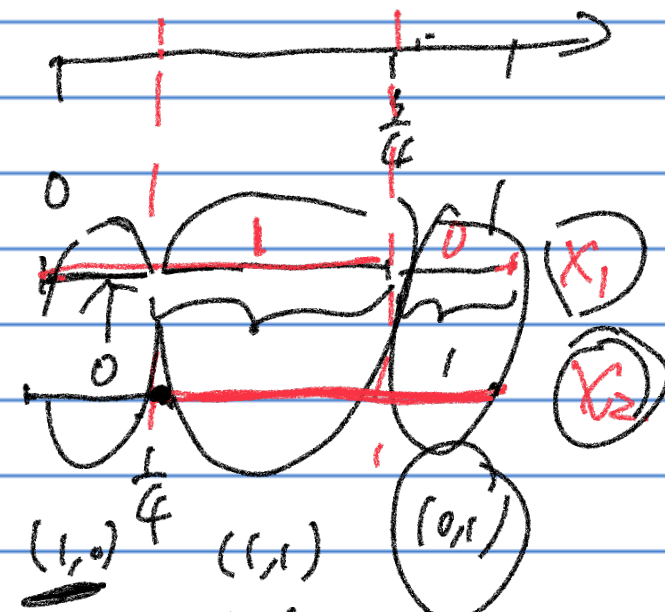
plans:

1. Concepts of Conditional P.D.F.;
   Independence
2. $E(E(X|Y)) = E(X)$ ✓
3. $V(X) = E(V(X|Y)) + V(E(X|Y))$ ✓

Conditioned p. M. F. ✓

$x_2 \backslash x_1$	0	1	$p_{X_2}(x_2)$
0	0 ✓	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$p_{X_1}(x_1)$	$\frac{1}{4}$	$\frac{3}{4}$	

$\omega \sim \text{Unif}([0,1])$



$$p(x_1 \mid x_2) = \frac{p(x_1, x_2)}{p_{X_2}(x_2)} = \frac{p(x_1, x_2)}{\sum_{x_1} p(x_1, x_2)} \checkmark$$

$$p(x_2 \mid x_1) = \frac{p(x_1, x_2)}{p_{X_1}(x_1)}$$

$X_2 = 0$

$p(x_1 \mid x_2 = 0)$:

x_1	0	1
p	0	1

$\neq$

$p(x_1 \mid x_2 = 1)$:

x_1	0	1
p	$\frac{1}{3}$	$\frac{2}{3}$

sum $\rightarrow 1$

$p(x_2 \mid x_1 = 0)$:

x_2	0	1
p	0	1

dependent

$p(x_2 \mid x_1 = 1)$:

x_2	0	1
p	$\frac{1}{3}$	$\frac{2}{3}$

Definition of Independence

X_1 and X_2 are indep if

$$p(x_1, x_2) = p_{X_1}(x_1) \cdot p_{X_2}(x_2)$$

for all x_1, and x_2.

$X_1 = x_1 \perp X_2 = x_2$

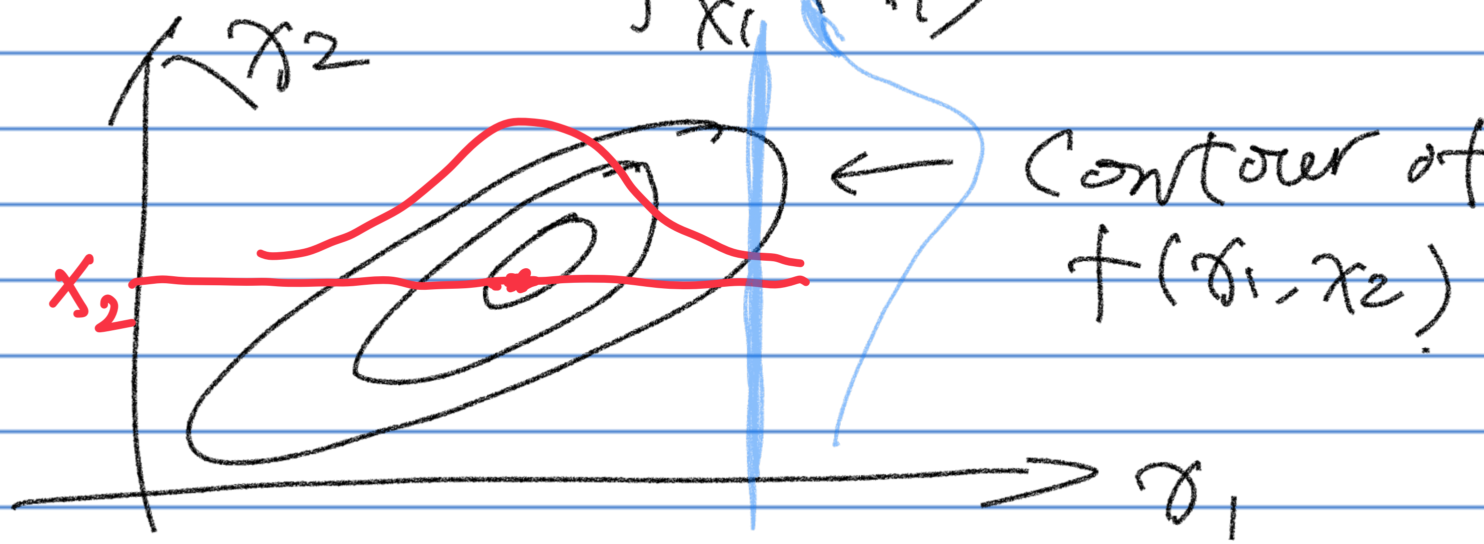
$$\iff \quad p(x_1 \mid x_2) = p_{X_1}(x_1), \text{ for all } x_1, x_2$$

$$\iff \quad p(x_2 \mid x_1) = p_{X_2}(x_2), \text{ for all } x_1, x_2$$

Conditional P.D.F.

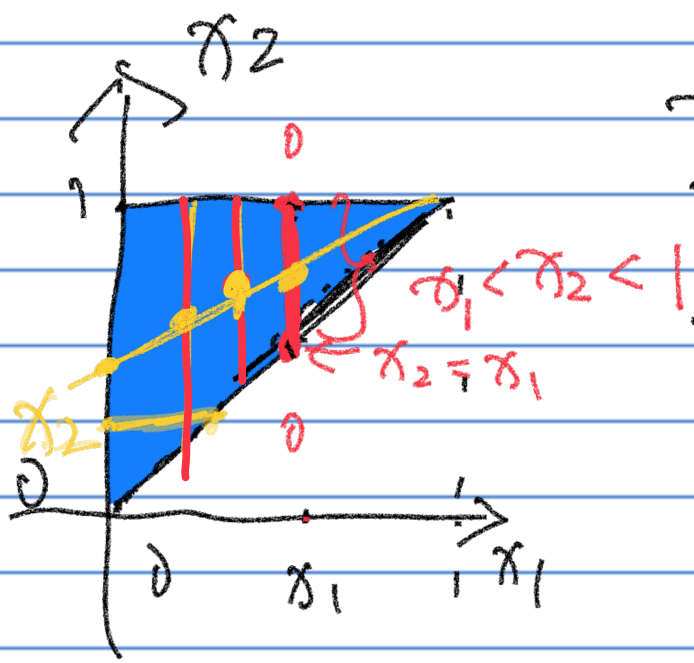
$$f(x_1 \mid x_2) = \frac{f(x_1, x_2)}{f_{X_2}(x_2)}$$

$$f(x_2 \mid x_1) = \frac{f(x_1, x_2)}{f_{X_1}(x_1)}$$



Example (e.g. 2.3.1.)

$$f(x_1, x_2) = \begin{cases} 2, & \text{if } 0 < x_1 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

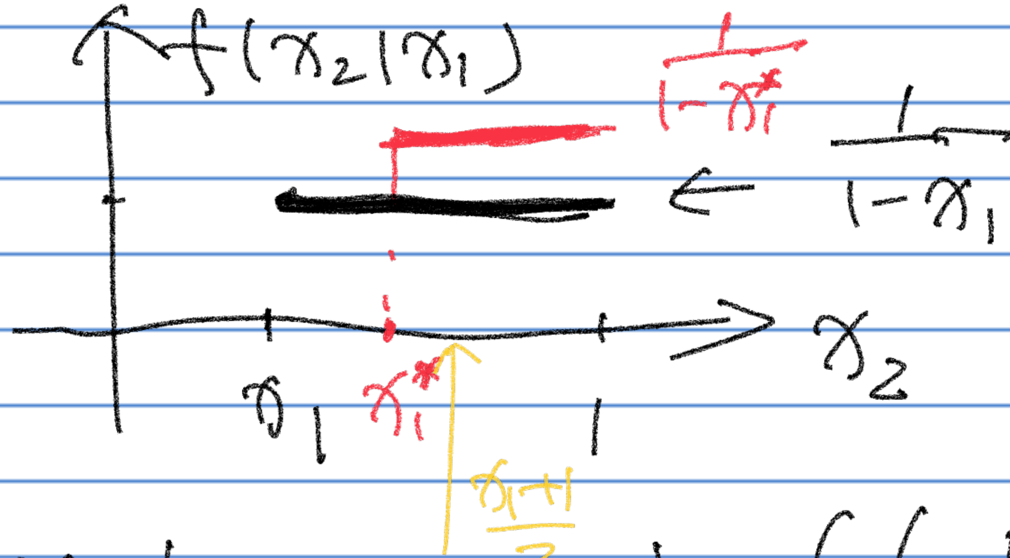


$$\Big\} \; 1 - x_1$$

$$f(x_2 \mid x_1) = \frac{f(x_1, x_2)}{f_{x_1}(x_1)}$$

$$= \frac{2}{\int_{x_1}^{1} 2 \, dx_2}, \quad \text{if } x_1 < x_2 < 1$$

$$= \frac{2}{2 \times (1 - x_1)} = \frac{1}{1 - x_1},$$



$$X_2 \mid x_1 \sim \mathrm{Unif}([x_1, 1])$$

$$X_1 \mid x_2 \sim \mathrm{Unif}([0, x_2])$$

Conditional expectation & Variance.

$$E(X_1 | x_2) = \int x_1 \cdot f(x_1 | x_2)\, dx_1$$

$$E(g(X_1) | x_2) = \int g(x_1) \cdot f(x_1 | x_2)\, dx_1$$

$$V(X_1 | x_2) = E\left[\underbrace{\left[X_1 - E(X_1 | x_2) \right]^2}_{g(X_1)} \Big| x_2 \right]$$

Example:

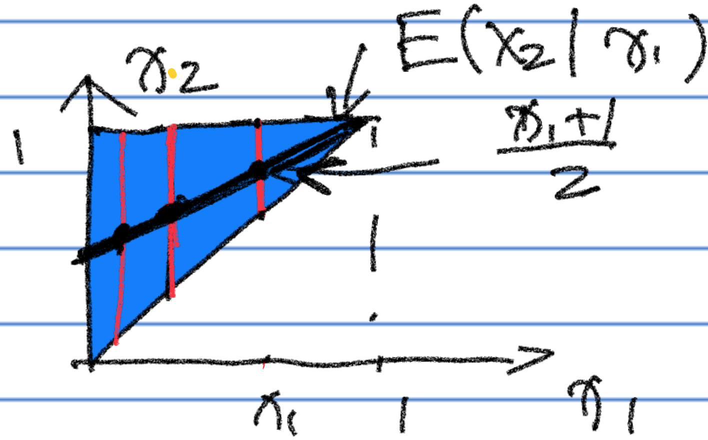
$$X_2 \mid x_1 \sim \text{unif}([x_1, 1])$$

$$E(X_2 \mid x_1) = \int_{-\infty}^{+\infty} x_2 \cdot f(x_2 \mid x_1)\, dx_2$$

$$= \int_{x_1}^{1} x_2 \frac{1}{1-x_1} dx_2$$

$$= \frac{1}{1-x_1} \cdot \frac{x_2^2}{2} \Big|_{x_2 = x_1}^{1}$$

$$= \frac{1}{1-x_1} \cdot \frac{1-x_1^2}{2} = \frac{1+x_1}{2}$$



$$E(X_2 | x_1) = \frac{x_1+1}{2} = g(x_1)$$

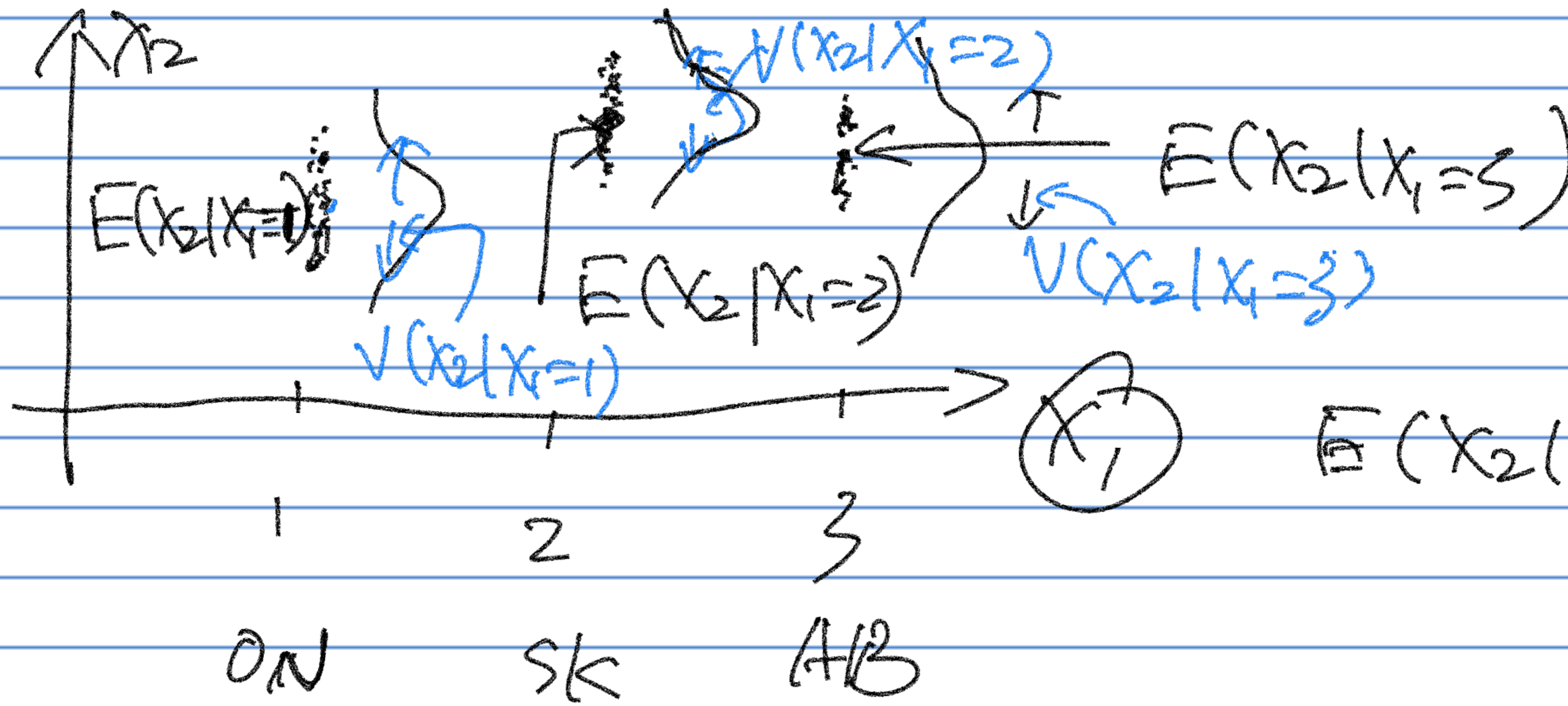
$$g(\cancel{X_1}) = \boxed{E(X_2 | \cancel{X_1})}$$

$E(X_2 | X_1)$ is a transformation of X_1

$V(X_2 \mid X_1)$ is a function of X_1

$V(X_2 \mid X_1)$ is a transformation of X_1 too, that is, is a R.V.

$$E\left[\underbrace{E(X_2 \mid X_1)}_{\text{a function of } X_1}\right] \qquad V(\underline{E(X_2 \mid X_1)}) \qquad E(\underbrace{V(X_2 \mid X_1)}_{\text{a fn of } X_1})$$



$E(X_2|X_1)$

X_2 : income of a person

Thm:

$$E(E(X_2 | X_1)) = E(X_2)$$

<u>Pf</u>: suppose X_1 & X_2 are continuous

$$E(X_2 | X_1) = \int x_2 \cdot f(x_2 | x_1)\, dx_2$$

$$= \int x_2 \cdot \frac{f(x_1, x_2)}{f_{X_1}(x_1)}\, dx_2$$

$$E(E(X_2 | X_1))$$

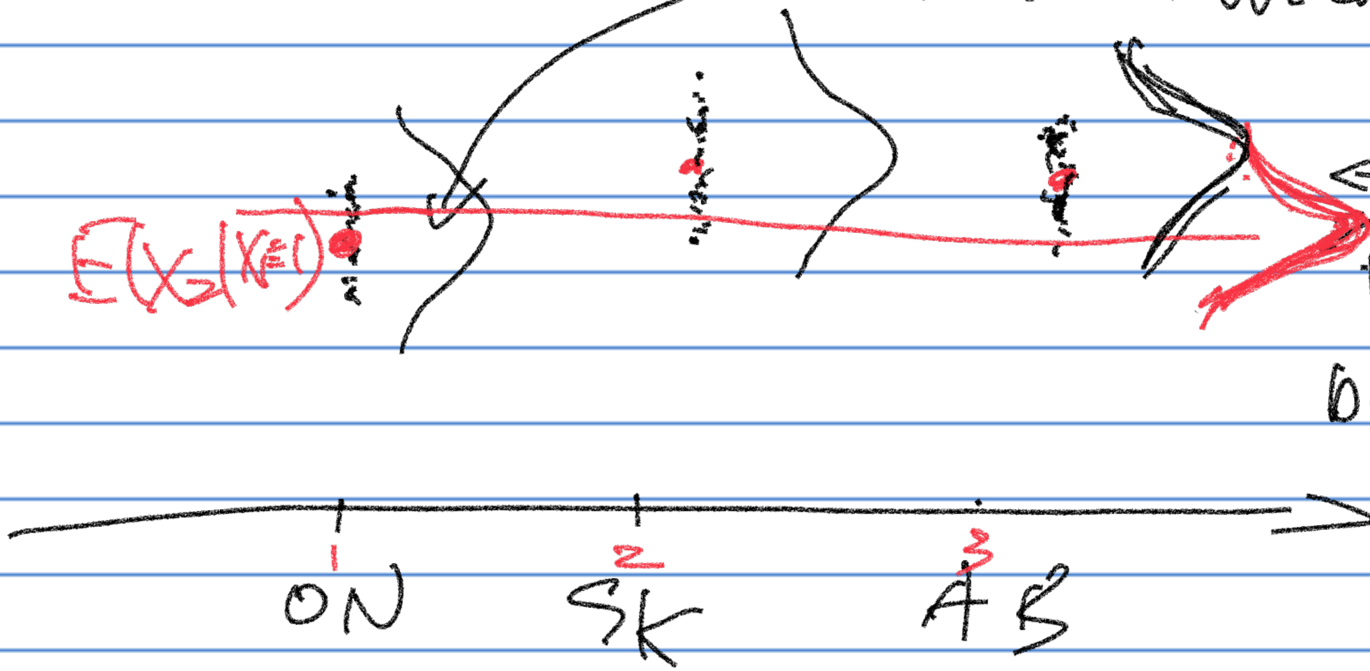
$$= \int \cancel{f_{X_1}(x_1)} \cdot \frac{1}{\cancel{f_{X_1}(x_1)}} \int x_2 \cdot f(x_1, x_2)\, dx_2\, dx_1$$

$$= \iint x_2 \cdot f(x_1, x_2)\, dx_2\, dx_1 = E(X_2)$$

Thm: Variance decomposition

$$V(X_2) = E(V(X_2|X_1)) + V(E(X_2|X_1))$$

Variance within groups



Variance between groups

X_1

Pf:

Let $h(X_1) = E(X_2 | X_1)$, $u = E(X_2)$

$$u = E(X_2) = E(E(X_2 | X_1))$$
$$= E(h(X_1))$$

$$V(X_2) = E((X_2 - u)^2)$$
$$= E((X_2 - h(X_1) + h(X_1) - u)^2)$$

$$= E_{X_1}\left[E_{X_2}\left((X_2 - h(X_1) + h(X_1) - u)^2 \mid X_1 \right) \right]$$

$h(X_1) \leftarrow E(X_2 \mid X_1)$

$$= E_{X_1}\Big[E_{X_2}\left((X_2 - h(X_1))^2 \mid X_1 \right) + \qquad \leftarrow V(X_2 \mid X_1)$$

$$\cancel{E_{X_2}}\left((h(X_1) - u)^2 \mid X_1 \right) + \qquad (h(X_1) - u)^2$$

Not a fn of X_2 $\rightarrow 0$

$$2\, E_{X_2}\left((X_2 - h(X_1))(h(X_1) - u) \mid X_1 \right) \Big]$$

$E(X_2 \mid X_1)$

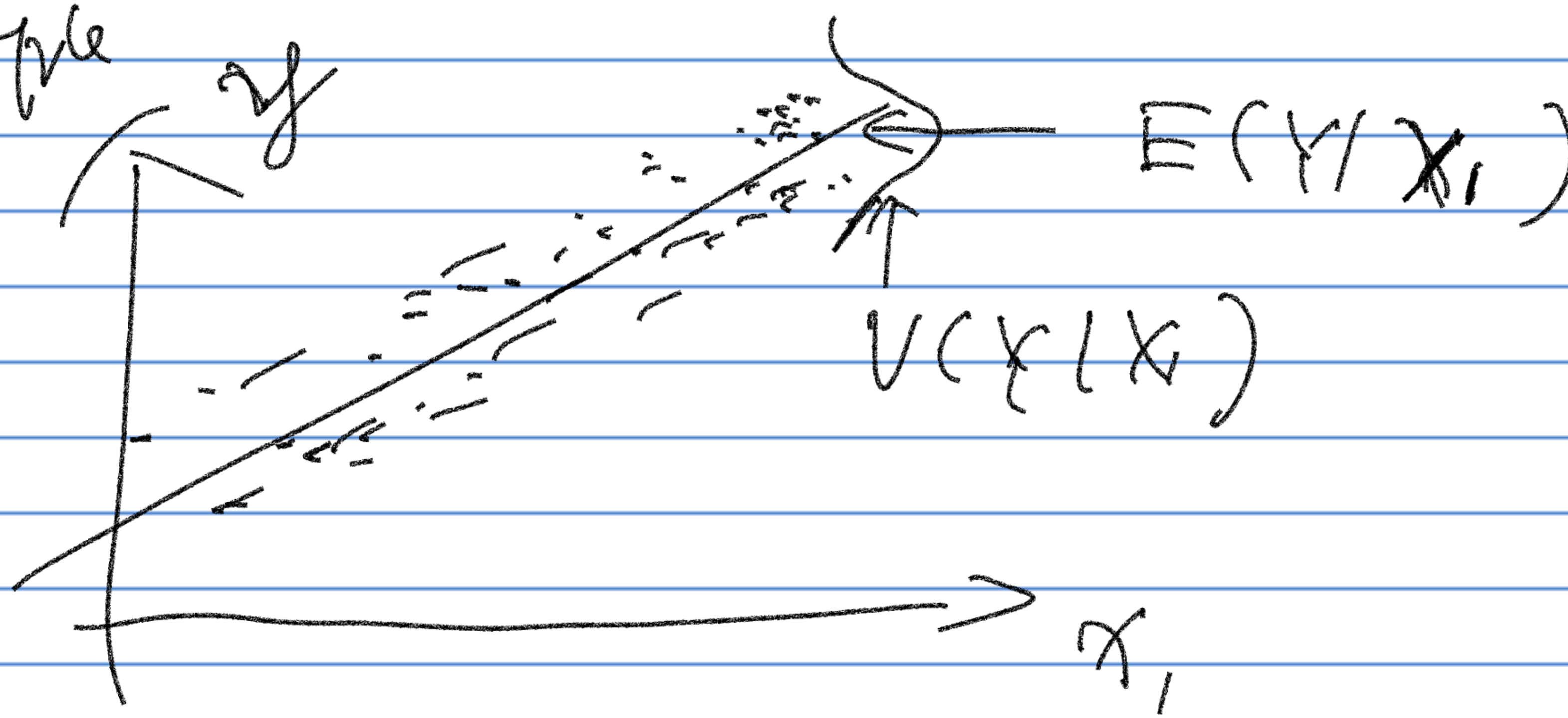
$$= E_{X_1}\left(V(X_2 \mid X_1) \right) + E_{X_1}\left((h(X_1) - u)^2 \right)$$

$h(X_1)$

$$= E\left(V(X_2 \mid X_1) \right) + V\left(E(X_2 \mid X_1) \right)$$

$$V(h(X_1)) = E\left((h(X_1) - u_h)^2 \right)$$

Example



$$V(Y) = V\left(E(Y|X_1)\right) + E\left(V(Y|X_1)\right)$$

Corralary :

$$V(X_2) \geq V(E(X_2 | X_1))$$

Rao - Blackwellization method for reducing Variance.