Lecture 14

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plan:

Sec 2-6, 2-7

1. extension to ≥ 3 random Variables.

2. Linear Combination (Sec 2-8)

≥ 3 Random variables (S2.6)

Continuous Random Variables

$f(x_1, x_2, \cdots, x_n)$ is a joint P.D.F.

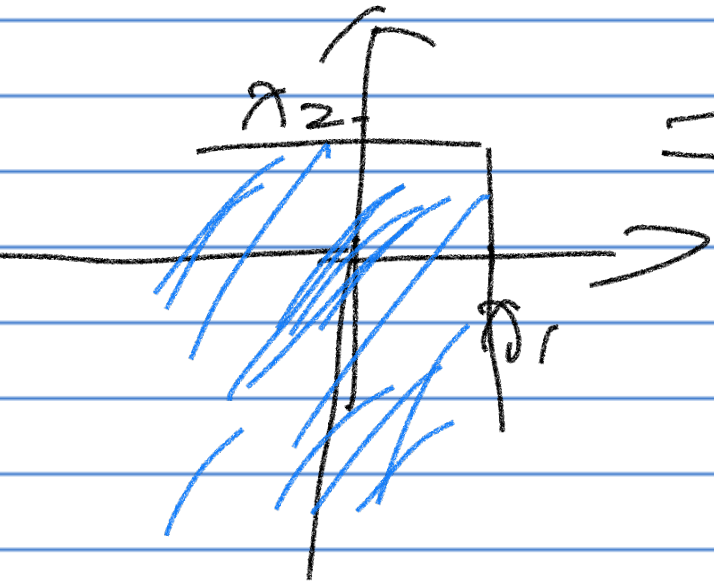
of $X_1, \cdots, X_n$ if

$$\int \cdots \int_A f(x_1, \cdots, x_n)\, \underline{dx_1 \cdots dx_n}$$

$$= P\left((X_1, \cdots X_n) \in A \right)$$

Joint C.D.F.

$$F(x_1, \cdots, x_n) = P(X_1 \le x_1, \cdots, X_n \le x_n)$$



$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(x_1, \cdots, x_n) \cdot dx_1 \cdots dx_n$$

Marginal P.D.F.

$s < n$

$$f(x_1, \dots, x_s)$$

$$= \int_{x_{s+1}} \cdots \int_{x_n} f(x_1, \dots, x_s, x_{s+1}, \dots, x_n)\, dx_{s+1} \cdots dx_n$$

Conditional P.D.F.

$$f(x_{s+1}, \dots, x_n \mid x_1, \dots, x_s) = \frac{f(x_1, \dots, x_n)}{f(x_1, \dots, x_s)}$$

Generally, think X_1, X_2 as random Vectors

Joint M.G.F.

$$M(t_1, \dots, t_n) = E\left(e^{t_1 X_1 + \dots + t_n X_n} \right)$$

Expectation

$$E\left(g(X_1, \dots, X_n) \right) = \int \cdots \int g(x_1, \dots, x_n) \cdot f(x_1, \dots, x_n) \, dx_1 \dots dx_n$$

Independence of $X_1, \ldots, X_n$:

We say $X_1, X_2, \ldots, X_n$ are indep

if

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

for all $x_1, x_2, \ldots, x_n$

Def of Random Sample (IID)

$X_1, X_2, \cdots, X_n$ are independent and identically distributed, i.e.

$$f(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} g(x_i)$$

where $g(x_i)$ is a P.D.F. of X_i.

Thm: M.G.F. of Sum of Radom Sample

Suppose $X_1, \ldots, X_n$ are IID.

let $T = \sum_{i=1}^{n} X_i$ (sample total)

then $M_T(t) = \left[M_{X_i}(t)\right]^n$ ✓

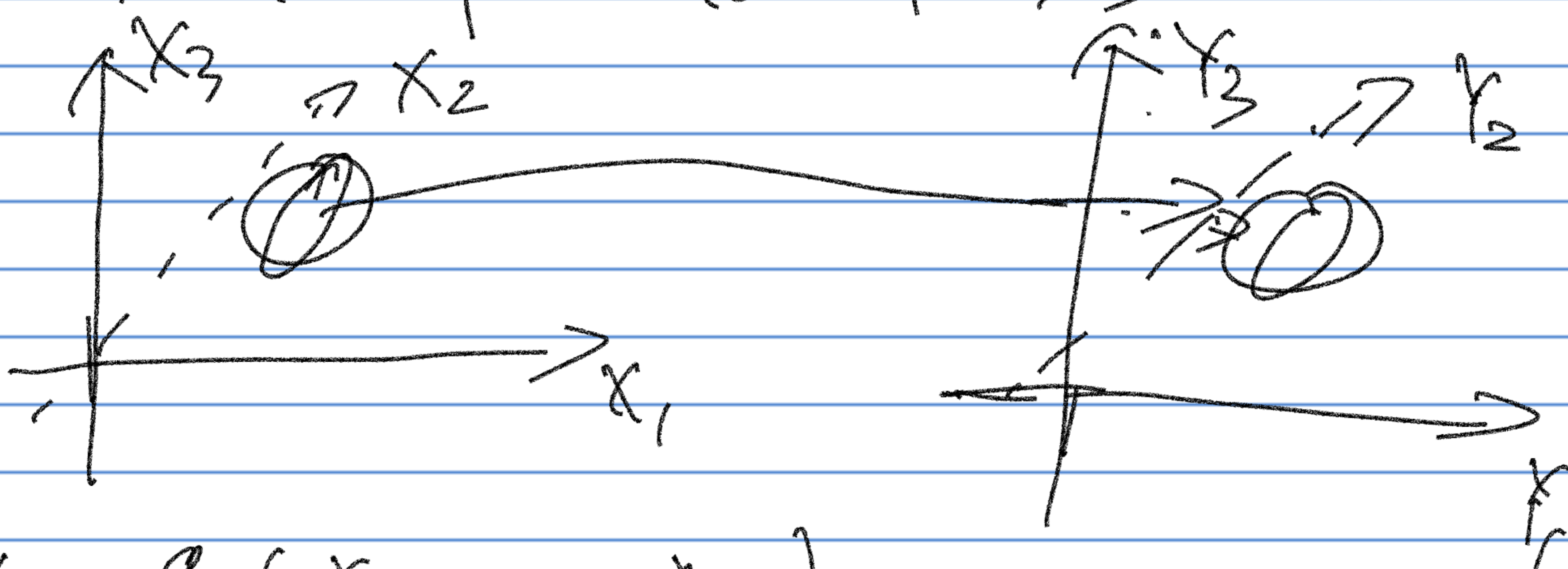
~~Pf~~ $M_T(s) = E(e^{sT}) = E(e^{s\sum_{i=1}^{n} X_i})$

$= E(e^{sX_1} \cdot e^{sX_2} \cdots e^{sX_n})$

$= E(e^{sX_1}) \cdots E(e^{sX_n})$

$= M_{X_1}(s) \cdots M_{X_n}(s) = \left[M_{X_i}(s)\right]^n$

Sec 2.7. Transformation of ≥ 3 r.v.



$$\begin{cases} Y_1 = g_1(X_1, \cdots, X_n) \\ \\ Y_n = g_n(X_1, \cdots, X_n) \end{cases}$$

1 — 1

$$\begin{cases} x_1 = t_1(Y_1, \dots, Y_n) \\ \\ x_n = t_n(Y_1, \dots, Y_n) \end{cases}$$

$$f_Y(y_1, \dots, y_n) = f_X(x_1, \dots, x_n) \cdot J$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1}, & \dots, & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1}, & \dots, & \frac{\partial x_2}{\partial y_n} \\ \dots & & \dots \\ \frac{\partial x_n}{\partial y_1}, & \dots, & \frac{\partial x_n}{\partial y_n} \end{vmatrix} +$$

Jacobian matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$$

$$+ 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$\begin{vmatrix} x_1 & x & x \\ 0 & x_2 & x \\ 0 & 0 & x_3 \end{vmatrix} = x_1 x_2 x_3$$

Linear Combination of Radom Variables

$(X_1, \dots, X_n)$ is a random vector.

let $T = \sum_{i=1}^{n} a_i X_i = a_1 X_1 + a_2 X_2 + \dots + a_n X_n$

Thm: suppose $E(|X_i|) < \infty$ (No assumption of independence)

$$T = \sum_{i=1}^{n} a_i X_i ,$$

then $E(T) = \sum_{i=1}^{n} a_i E(X_i)$. linearity of E.

pf: $E(T) = \int \cdots \int \left(\sum_{i=1}^{n} a_i x_i \right) f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n$

$$= \sum_{i=1}^{n} \underline{\int \cdots \int} \, a_i \, \underline{x_i f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n}$$

$$= \sum_{i=1}^{n} a_i \cdot E(X_i)$$

Thm:

$$T = \sum_{i=1}^{n} a_i X_i, \quad W = \sum_{j}^{m} b_j Y_j$$

$$Cov(T, W) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$$

Pf: Let $\mu_i = E X_i$, $\nu_j = E(Y_j)$

$$Cov(T, W) = E\left((T - E(T)) \cdot (W - E(W)) \right)$$

$$= E\left(\left[\sum_{i=1}^{n} a_i (X_i - \mu_i) \right] \left[\sum_{j=1}^{m} b_j (Y_j - \nu_j) \right] \right)$$

$$= E\left(\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \underline{(X_i - \mu_i)(Y_j - \nu_j)} \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$$

Thm:

$$T = \sum_{i=1}^{n} a_i X_i$$

$$V(T) = \sum_{i=1}^{n} a_i^2 V(X_i) + 2 \sum_{i<j} a_i a_j \operatorname{Cov}(X_i, X_j)$$

Pf:

$$V(T) = \operatorname{Cov}(T, T)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \operatorname{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} a_i^2 V(X_i) + 2 \sum_{i<j} a_i a_j \operatorname{Cov}(X_i, X_j)$$

Thm: suppose $X_1, \ldots, X_n$ are independent.

Let $T = \sum_{i=1}^{n} a_i X_i$

Then $V(T) = \sum_{i=1}^{n} a_i^2 V(X_i)$

"linearity" of Variance when X_i's are indep.

Pf: when $X_i \perp X_j$, $Cov(X_i, X_j) = 0$.

Example:

$$V(\bar{X}_1 - \bar{X}_2) = V(\bar{X}_1) + V(\bar{X}_2)$$

if $\bar{X}_1$ and $\bar{X}_2$ indep

$$V(\bar{X}_1) = V\left(\sum_{i=1}^{n} X_i / n\right)$$

$$= \frac{1}{n^2} \cdot V\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2} \cdot \sum_{i=1}^{n} 1^2 V(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

where $V(X_i) = \sigma^2$

Example:

$X_1, \cdots, X_n$ is a random sample, i.e.

$X_1, \cdots, X_n$ are IID.

Let $\mu = E(X_i)$, $\sigma^2 = V(X_i)$.

Let $$\bar{X} = \frac{X_1 + \cdots + X_n}{n}$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Then: $E(\bar{X}) = \mu$, $E(S^2) = \sigma^2$, $V(\bar{X}) = \frac{\sigma^2}{n}$

Pf:

$$E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$$

$$= \frac{1}{n}\cdot n\cdot u = u, \quad V(\overline{X}) = \frac{\sigma^2}{n}$$

$$E(S^2) = E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2\right)$$

$$= E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i^2 - 2\overline{X}\cdot X_i + \overline{X}^2)\right)$$

$$= E\left(\frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right]\right)$$

$$= \frac{1}{n-1}\left[n(u^2+\sigma^2) - n\cdot\left(u^2+\frac{\sigma^2}{n}\right)\right]$$

$$= \frac{1}{n-1}\left[(n-1)\sigma^2\right] = \sigma^2$$