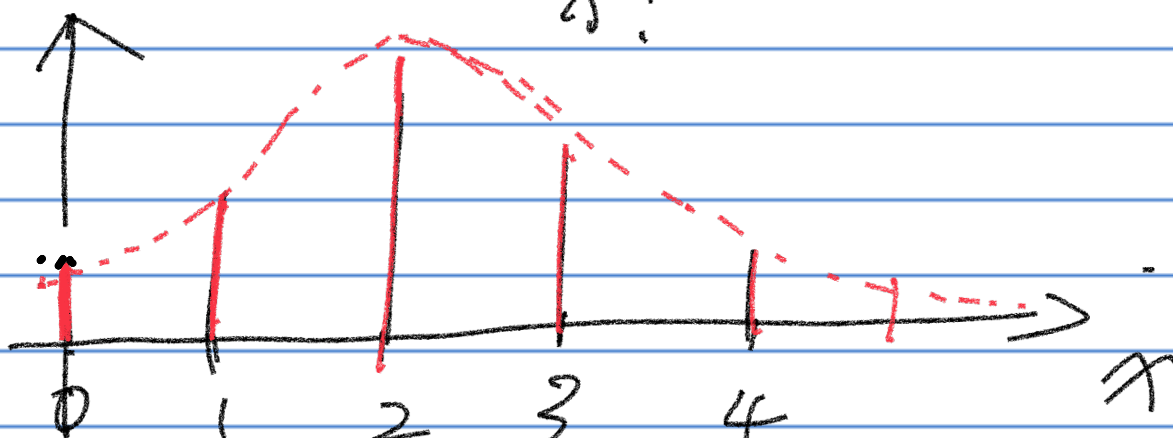
Lecture 16

Longhai Li, Nov 2, 2021

poisson distribution.

$X \geq 0$

$$p(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}, \quad \text{for } x = 0, 1, 2, \ldots$$



Law of small numbers

$X \sim \text{poisson}(\lambda)$ [$Po(\lambda)$]

$$E(X) = \sum_{x=1}^{+\infty} \frac{e^{-\lambda} \lambda^{x}}{x!} x$$

$$= e^{-\lambda} \left[\lambda \sum_{x=1}^{+\infty} \frac{\lambda^{x-1}}{(x-1)!} \right]$$

$$= e^{-\lambda} \left[\lambda \sum_{j=0}^{+\infty} \frac{\lambda^{j}}{j!} \right]$$

$$= e^{-\lambda} \left[\lambda e^{\lambda} \right] = \lambda$$

$V(X) = \lambda$

$M_X(t) = e^{\lambda(e^t - 1)}$

$Pois(\lambda) \approx Binomial(n, p)$, when p is small, where $\lambda = np$

$$n \cdot p_n \to \lambda$$

$$b(x; n, p)$$

$$= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \frac{1}{x!} \underbrace{n \times (n-1) \times \cdots \times (n-x+1)}_{x \text{ terms}} p^x (1-p)^n (1-p)^{-x}$$

$$\approx \frac{1}{x!} n^x p^x \left(1 - \frac{np}{n}\right)^n (1-p)^{-x}$$

$$= \frac{1}{x!} \lambda^x e^{-\lambda}$$

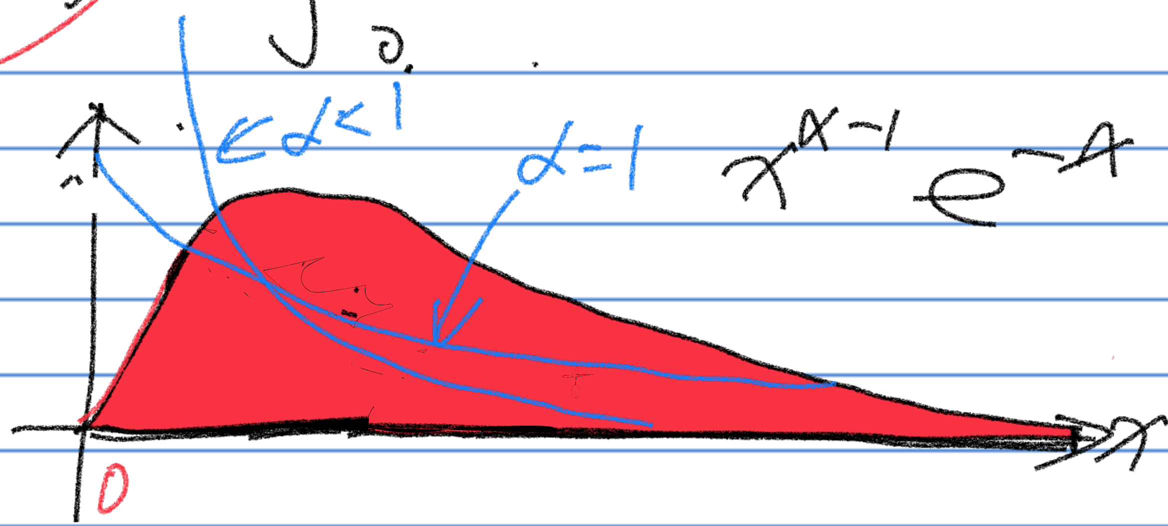
$\left(1 + \frac{a}{n}\right)^n \to e^a$ $\qquad p \to 0$ $\qquad 1$

This approx. is good when n is large and p is small, Law of Small #.

Gamma distribution

Gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$



Some properties of $\Gamma(\alpha)$

(1) $\Gamma(1) = 1$

$$\int_0^{+\infty} e^{-x}\,dx = 1$$

2) $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

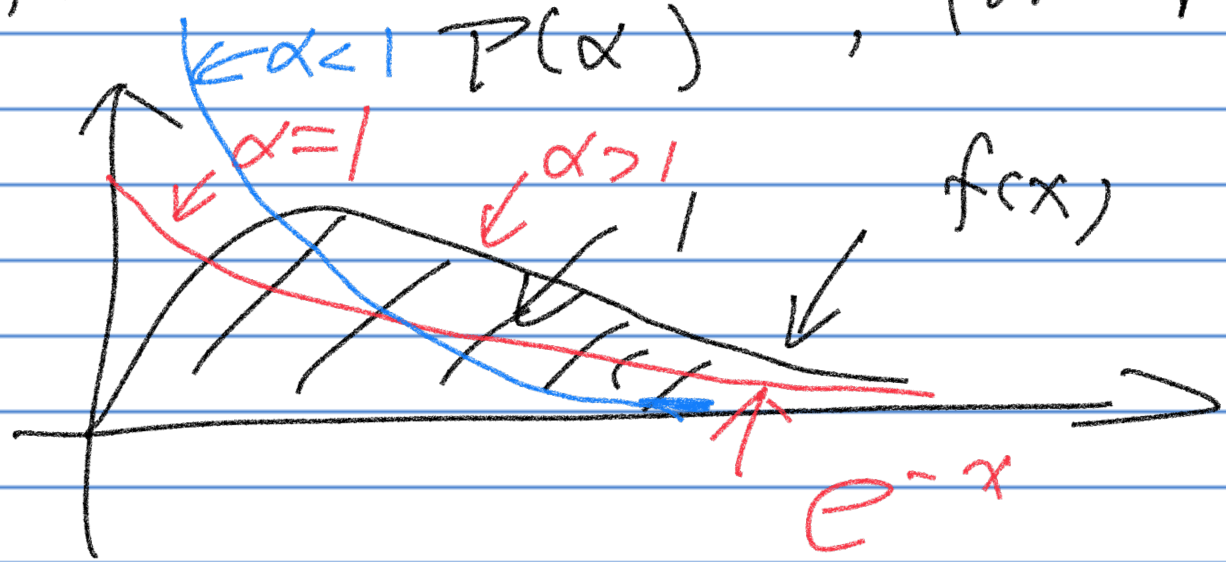
$$\begin{aligned}\Gamma(n+1) &= n \cdot \Gamma(n) \\ &= n \cdot (n-1)\, \Gamma(n-1) \\ &= n \cdot (n-1) \cdot \cdots 2 \cdot \Gamma(1) \\ &= n!\end{aligned}$$

3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Gamma dist. (Standard), Gamma(α, 1)

p.d.f.

$$f(x) = \frac{e^{-x}\, x^{\alpha-1}}{\Gamma(\alpha)}, \quad \text{for } x \geq 0$$



$X \sim \text{Gamma}(\alpha, 1)$

$$E(X) = \int_0^{+\infty} x \cdot \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} \, dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} x^{\overset{\alpha'}{(\alpha+1)}-1} e^{-x} \, dx$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \Gamma(\alpha+1) = \alpha$$

$$E(X^2) = \int_0^{+\infty} x^2 \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} \, dx$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}$$

$$= \frac{(\alpha+1) \cdot \alpha \cdot \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$$

$$= \alpha^2 + \alpha$$

$$V(X) = E(X^2) - [E(X)]^2 = \alpha^2 + \alpha - \alpha^2$$

$$= \alpha$$

$$M_X(t) = E(e^{tX})$$

$$= \int_0^{+\infty} e^{tx} \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} dx$$

$$= \int_0^{+\infty} \frac{x^{\alpha-1} e^{-(1-t)x}}{\Gamma(\alpha)} dx \quad = y$$

Let

$y = (1-t)x$

$dx = \frac{1}{1-t} dy$

$x = \frac{y}{1-t}$

$$= \int_0^{+\infty} \frac{\left(\frac{1}{1-t}\right)^{\alpha-1} y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} \frac{1}{1-t} dy$$

$$= \left(\frac{1}{1-t}\right)^{\alpha} \cdot 1, \quad \text{for} \quad t < 1$$

$$E(X^k) = \int_0^{+\infty} \frac{x^k e^{-x} x^{\alpha-1}}{\Gamma(\alpha)} dx$$

$$= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

$E(X^k) < \infty$ for all $k \geq 0$

The tail of Gamma is not heavy.

$Y = X \cdot \beta$ $\quad X \sim \mathrm{Gamma}(\alpha, 1)$

β is called scale parameter.

α is called shape parameter.

$\lambda = \frac{1}{\beta}$ is called rate parameter.

$Y \sim \mathrm{Gamma}(\alpha, \beta)$

P.D.F.

$$Y = X \cdot \beta$$

$$f_Y(y) = f_X\left(\frac{y}{\beta}\right) \cdot \left|\frac{dx}{dy}\right|$$

$$= f_X\left(\frac{y}{\beta}\right) \cdot \frac{1}{\beta}$$

$$= \frac{e^{-\frac{y}{\beta}} \cdot \left(\frac{y}{\beta}\right)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{1}{\beta}$$

$$= \frac{e^{-\frac{y}{\beta}} \cdot y^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{1}{\beta^{\alpha}}, \text{ for } y > 0$$

P.D.F. with $\lambda = \frac{1}{\beta}$.

$$f(y) = \frac{e^{-\lambda y}\, y^{\alpha-1}}{\Gamma(\alpha)} \cdot \lambda^{\alpha}, \text{ for } y \geq 0$$

$Y = X \cdot \beta$, $X \sim \text{Gamma}(\alpha, 1)$.

$E(Y) = E(X \cdot \beta) = \beta \cdot E(X) = \beta \cdot \alpha$

$V(Y) = \beta^2 \cdot V(X) = \beta^2 \cdot \alpha$

$$M_Y(t) = E(e^{tY})$$

$$= E(e^{t \cdot X\beta}) = M_X(t\beta)$$

$$= (1 - t\beta)^{-\alpha}.$$

$$= \frac{1}{(1-t\beta)^{\alpha}}$$

or

$$= \frac{1}{\left(1 - \frac{t}{\lambda}\right)^{\alpha}}$$