Lecture 17

Longhai Li, Nov 4,2021

Plan:

1) More on Gamma

2) Normal distribution.

Special Gamma.

1) $\alpha=1,\ \beta=1$

$$f(x)=\frac{e^{-x}\,x^{\alpha-1}}{\Gamma(\alpha)}=e^{-x}\ \text{for } x>0$$

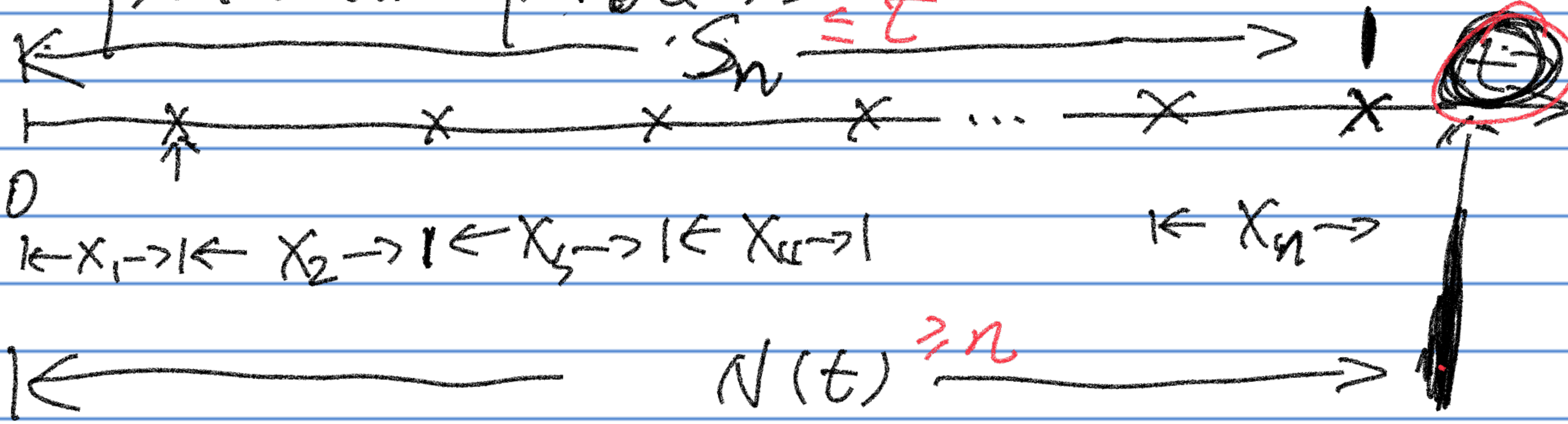
$$\text{Gamma}(\alpha=1,\ \beta=1)=\exp(1)$$

$$\text{Gamma}(\alpha=1,\ \beta)=\exp\left(\frac{1}{\beta}\right)$$

$(\ \ )$

2) Erlang distribution $\alpha = n$.

Poisson process



$$\begin{cases} N(t) = \#\text{ of events btw } [0, t] \\ S_n = X_1 + X_2 + \cdots + X_n \end{cases}$$

$N(t)$ takes values 0, 1, 2, ...

S_n is continuous, $S_n \geq 0$

This process is called poisson process with rate λ if

$$N(t) \sim \text{poisson}(\lambda t) \quad \checkmark$$

and something more.

$$E(N(t)) = \lambda t$$

$$E(N(1)) = \lambda$$

Thm:

$N(t) \sim \text{poisson}(\lambda t)$

$\Updownarrow$ $S_n = X_1 + \cdots + X_n \sim \text{Gamma}(n, \frac{1}{\lambda})$.

$X_1, \cdots, X_n$ are IID $\exp(\lambda)$

Pf:

$F_{S_n}(t) = P(S_n \le t) = P(N(t) \ge n)$

$$P(S_n \le t) = P(N(t) \ge n)$$

$$= \sum_{k=n}^{+\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$F'_{S_n}(t) = \left[\frac{d}{dt} \sum_{k=n}^{+\infty} \cdot \frac{e^{-\lambda t}(\lambda t)^k}{k!}\right]$$

$$= \sum_{k=n}^{+\infty} \frac{e^{-\lambda t}(-\lambda)(\lambda t)^k + e^{-\lambda t}(\lambda t)^{k-1} \cdot k \cdot \lambda}{k!}$$

$(k-1)!$

$$= \lambda \cdot \left[- \sum_{k=n}^{+\infty} \frac{e^{-\lambda t} \cdot (\lambda t)^k}{k!} + \sum_{k=n}^{+\infty} \frac{e^{-\lambda t} (\lambda t)^{k-1}}{(k-1)!} \right]$$

$$= \lambda \left[- \sum_{k=n}^{+\infty} \frac{e^{-\lambda t} (\lambda t)^k}{k!} + \sum_{k=n-1}^{+\infty} \frac{e^{-\lambda t} (\lambda t)^k}{k!} \right]$$

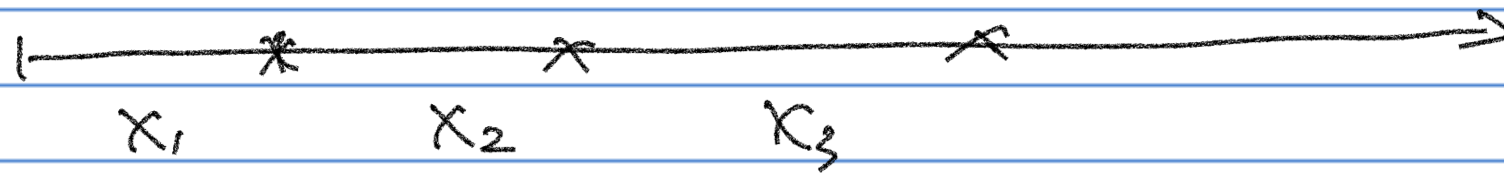
$$= \lambda \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}$$

This is the P.D.F. of Gamma$(n, \beta = \frac{1}{\lambda})$

$S_n \sim \text{Gamma}(n, \beta = \frac{1}{\lambda})$

Erlang distribution

$S_1 = X_1 \sim \text{Gamma}(1, \frac{1}{\lambda}) = \exp(\lambda)$



$X_1, X_2, \ldots, X_n \overset{IID}{\sim} \exp(\lambda)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

$\quad\quad\quad\quad\quad\quad\quad\quad \text{Gamma}(1, \beta = \frac{1}{\lambda})$

$S_n = X_1 + \cdots + X_n$

sum of n IID $\exp(\lambda)$ r.v.

Thm: additivity of Gamma

If $X_i \sim \text{Gamma}(\underline{\alpha_i}, \beta)$,

$X_1, \dots, X_n$ are independent.

then

$\underline{\underline{X_1 + \cdots + X_n}} \sim \text{Gamma}\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$

$\underline{\text{Pf}}$: $M_{X_i}(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha_i}$

$$M_{X_1 + \cdots + X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \left(1 - \frac{t}{\beta}\right)^{-\alpha_i}$$

$$= \left(1 - \frac{t}{\beta}\right)^{-\sum_{i=1}^{n} \alpha_i}$$

This is the M.G.F. of $\text{Gamma}\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$

Additivity of Binomial.

If $X_i \sim Bin(n_i, p)$, indep.

then $S_n = X_1 + \cdots + X_n \sim Bin\left(\sum_{i=1}^{k} n_i, p\right)$.

Pf: $M_{X_i}(t) = (q + pe^t)^{n_i}$

$$M_{S_n}(t) = \prod_{i=1}^{k} M_{X_i}(t)$$

$$= (q + pe^t)^{\sum_{i=1}^{k} n_i}$$

Additivity of poisson(λ).

If $X_i \sim$ poisson(λ_i), indep

for $i=1, \dots, n$

then $S_n = X_1 + \dots + X_n \sim$ poisson$\left(\sum_{i=1}^{n} \lambda_i\right)$

Pf: $M_{X_i}(t) = e^{\lambda_i(e^t - 1)}$

$M_{S_n}(t) = e^{\sum_{i=1}^{n} \lambda_i (e^t - 1)}$

Beta distribution.

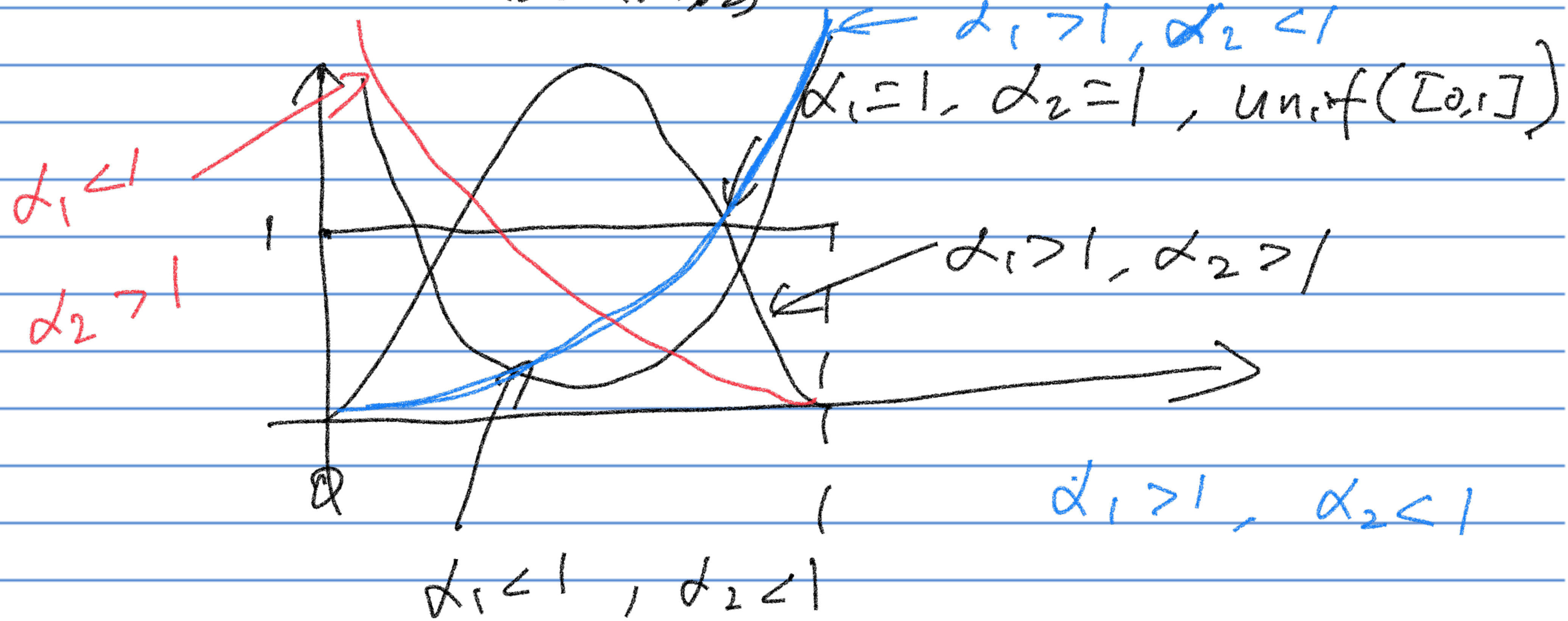
$X \sim \text{Gamma}(\alpha_1, 1)$

$Y \sim \text{Gamma}(\alpha_2, 1)$

$$W = \frac{X}{X+Y} \sim \text{Beta}(\alpha_1, \alpha_2)$$

$W \in [0, 1]$

$$f(w) = \frac{1}{Beta(\alpha_1, \alpha_2)} w^{\alpha_1 - 1} (1-w)^{\alpha_2 - 1}, \quad \text{for } w \in [0,1]$$



$$E(X) = \frac{\alpha_1}{\alpha_1+\alpha_2}$$

$$V(X) = \frac{\alpha_1 \cdot \alpha_2}{(\alpha_1+\alpha_2+1)(\alpha_1+\alpha_2)^2}$$

A remark: $X \sim \text{Gamma}(\alpha_1, \beta)$, $\frac{X}{\beta} \sim \text{Gamma}(\alpha_1, 1)$

$Y \sim \text{Gamma}(\alpha_2, \beta)$, $\frac{Y}{\beta} \sim \text{Gamma}(\alpha_2, 1)$

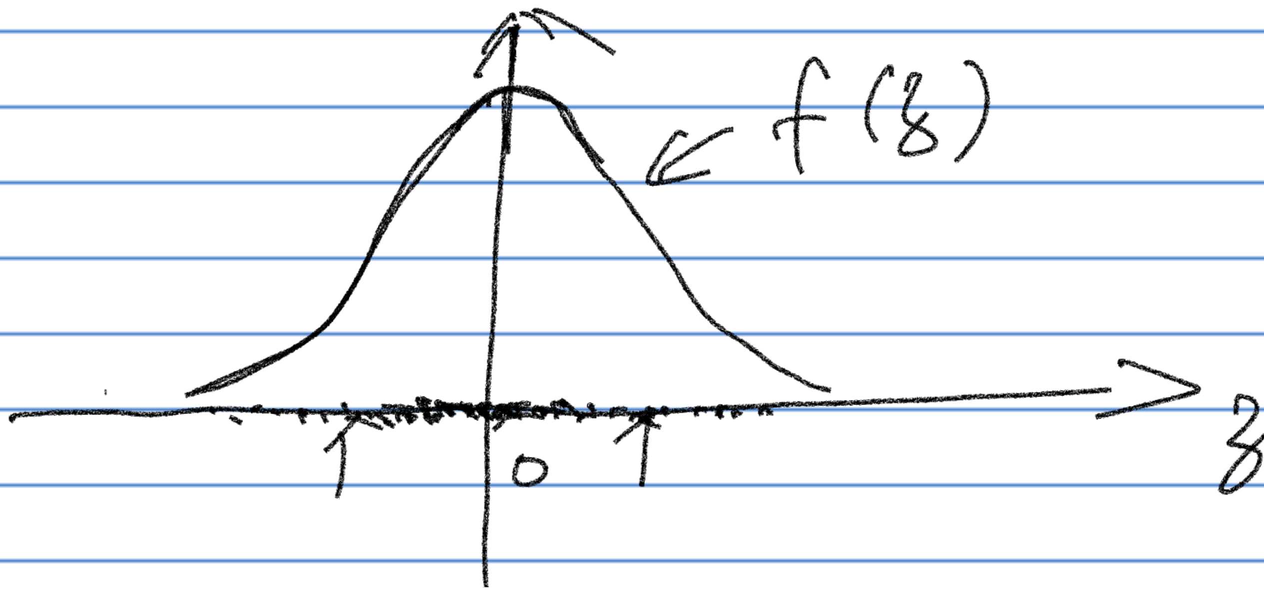
$$W = \frac{X}{X+Y} = \frac{\frac{X}{\beta}}{\frac{X}{\beta}+\frac{Y}{\beta}} \sim \text{Beta}(\alpha_1, \alpha_2)$$

Normal distribution

$Z \sim N(0,1)$:

P.D.F.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < +\infty$$



$E(Z) = 0$

$V(Z) = E(Z^2)$

$$= \int_{-\infty}^{+\infty} z^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= 2 \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} z^2 e^{-\frac{z^2}{2}} dz \qquad z = \sqrt{x}$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} x e^{-\frac{x}{2}} d\sqrt{x}$$

...

$$= 1$$

M.G.F.

$$M_Z(t) = E(e^{tZ})$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{tz} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[z^2 - 2tz + t^2 - t^2]} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[z-t]^2 + \frac{1}{2}t^2} dz$$

$$= e^{\frac{1}{2}t^2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(z-t)^2} dz, \quad z - t = x$$

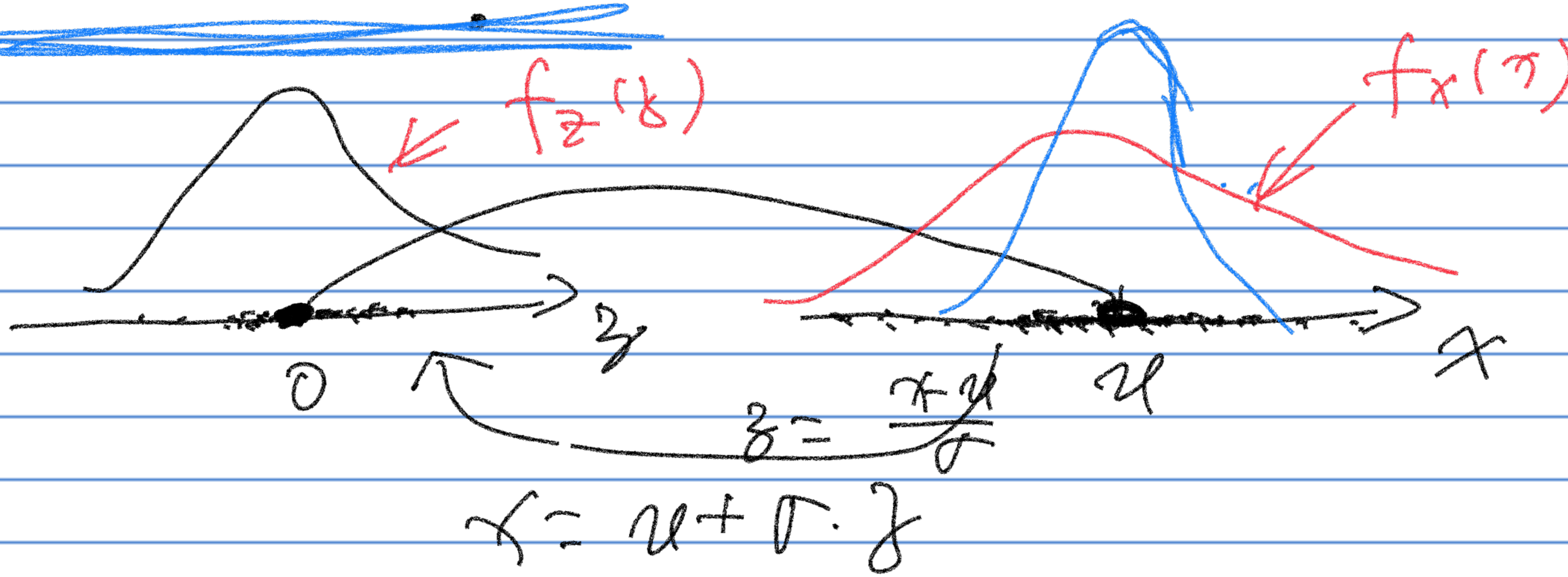
$$= e^{\frac{1}{2}t^2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx}_{1}$$

$$= e^{\frac{1}{2}t^2}$$

$$F_Z(z) = \int_{-\infty}^{z} f(z)\,dz$$, no closed form

z		$F(z)$
-3.0	0.00	
-3.1		

$$X = \mu + \sigma Z \,, \quad Z \sim N(0,1)$$



$$X \sim N(\mu, \sigma^2)$$

P.D.F. of X

$$Z = \frac{X-\mu}{\sigma}$$

$$f_X(x) = f_Z\left(\frac{x-\mu}{\sigma}\right)\left|\frac{dz}{dx}\right|$$

$$= f_Z\left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sigma}$$

$$E(X) = E(\mu + \sigma Z)$$

$$= \mu + \sigma \cdot E(Z)$$

$$= \mu$$

$$V(X) = V(\mu + \sigma Z)$$

$$= \sigma^2 \cdot V(Z) = \sigma^2$$

$$
\begin{aligned}
M_X(t) &= E\left(e^{tX}\right) \\
&= E\left(e^{t(\mu+\sigma Z)}\right) \\
&= e^{t\mu} \cdot E\left(e^{t\sigma Z}\right) \\
&= e^{t\mu} \cdot e^{\frac{1}{2}(t\sigma)^2} \\
&= e^{\mu t + \frac{1}{2}\sigma^2 \cdot t}
\end{aligned}
$$

Additivity of $N(\mu, \sigma^2)$

Thm:

$X_i \sim N(\mu_i, \sigma_i^2)$.

$X_1, \dots, X_n$ are indep.

$$X_1 + \dots + X_n \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

Pf:

$$M_{X_i}(t) = e^{u_i t + \frac{1}{2}\sigma_i^2 t^2}$$

$$M_{X_1+\dots+X_n}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$= e^{\sum_{i=1}^{n}(u_i t + \frac{1}{2}\sigma_i^2 t^2)}$$

$$= e^{\underbrace{\sum_{i=1}^{n} u_i}_{u} t + \frac{1}{2}\underbrace{\sum_{i=1}^{n}\sigma_i^2}_{\sigma^2} t^2}$$

This is the MGF of

$$N\left(\sum_{i=1}^{n} u_i, \sum_{i=1}^{n}\sigma_i^2\right)$$

[illegible]

$$X_1 \sim N(\mu_1, \sigma_1^2)$$

$$X_2 \sim N(\mu_2, \sigma_2^2)$$

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2).$$

Thm:

If $X_i \sim N(\mu_i, \sigma_i^2)$ indep.

then $\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$

Pf:

$$a_i X_i \sim N(a_i \mu_i, a_i^2 \sigma_i^2)$$

appling previous formula

Example:

$$X_1, \ldots, X_n \overset{IID}{\sim} N(\mu, \sigma^2)$$

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n} = \sum_{i=1}^{n} a_i X_i$$

where $a_i = \frac{1}{n}$

$$\sum_{i=1}^{n} a_i \mu_i = \sum \left(\frac{1}{n} \cdot \mu\right) = \mu$$

$$\sum_{i=1}^{n} a_i^2 \sigma_i^2 = \sum \frac{1}{n^2} \cdot \sigma^2 = \frac{\sigma^2}{n}$$

so $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$