Stat 342
Mathematical Statistics
Lecture 18

Longhai Li, November 16, 2021

Plan:

1. χ^2 distribution (Def & Rel. with Gamma)
2. Student's Theorem (Sampling dist. of $\bar{X}$ & S^2)
3. ~~t & F distribution~~.

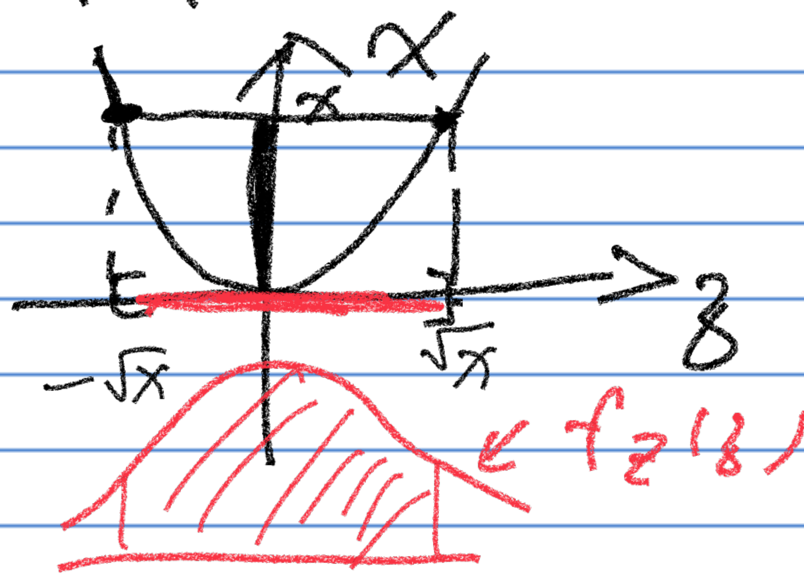
χ^2 distribution

Def:

Let $Z \sim N(0,1)$

$X = Z^2 \sim \chi^2_{1}$ degree freedom 1

PDF of X:



For $x \geq 0$

$X \leq x$

$\Leftrightarrow Z^2 \leq x$

$\Leftrightarrow -\sqrt{x} \leq Z \leq \sqrt{x}$

$$F_X(x) = P(X \le x) = P(-\sqrt{x} \le Z \le \sqrt{x})$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, d$$

$$\frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$$

For $x \ge 0$

$$f_X(x) = F_X'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} \cdot \frac{d\sqrt{x}}{dx} \quad z$$

$$- \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} \cdot \frac{d(-\sqrt{x})}{dx}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} \cdot \cancel{\frac{1}{2}} \cdot x^{\frac{1}{2}-1}$$

$$= \frac{1}{2^{\frac{1}{2}}\,\Gamma(\frac{1}{2})} \cdot x^{\frac{1}{2}-1} e^{-\frac{x}{2}}$$

α β

$$Z^2 \sim \text{Gamma}\left(\alpha = \frac{1}{2},\ \beta = 2\right)$$

$$\chi^2_1 = \text{Gamma}\left(\alpha = \frac{1}{2},\ \beta = 2\right)$$

$$E(Z^2) = \alpha \cdot \beta = \frac{1}{2} \cdot 2 = 1$$

$$\stackrel{\text{alt.}}{=} V(Z) = 1$$

$$V(Z^2) = \alpha \cdot \beta^2 = \frac{1}{2} \cdot 2^2 = 2$$

Def of χ^2_n

Let $Z_1, \ldots, Z_n \overset{IID}{\sim} N(0,1)$

Let $X = Z_1^2 + \cdots + Z_n^2$

We say $X \sim \chi^2_n$, chi square with n degrees of freedom

By additivity of Gamma,

$$X \sim \text{Gamma}\left(\tfrac{1}{2} + \tfrac{1}{2} + \cdots + \tfrac{1}{2}, 2\right)$$

$$= \text{Gamma}\left(\tfrac{n}{2}, 2\right).$$

$$f_X(x) \propto e^{-\frac{x}{2}} \cdot x^{\frac{n}{2}-1}$$

$$E(X) = \alpha \cdot \beta = \frac{n}{2} \cdot 2 = n.$$

Scaled χ_n^2:

$$Y = \frac{X}{n} = \frac{Z_1^2 + \dots + Z_n^2}{n}$$

$$E(Y) = \frac{E(X)}{n} = \frac{n}{n} = 1$$

$$V(X) = \alpha \cdot \beta^2 = \frac{n}{2} \cdot 2^2 = 2n$$

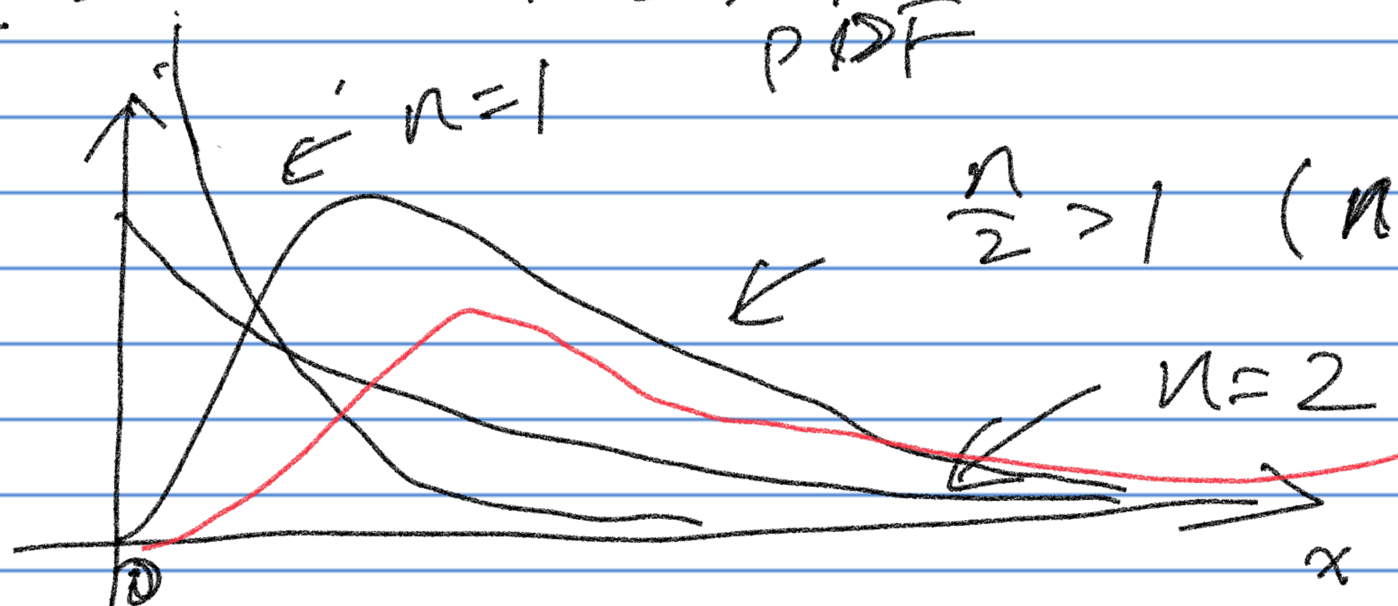
M.G.F. of X

$$M_X(t) = (1-2t)^{-\frac{n}{2}}$$

special χ^2_n:

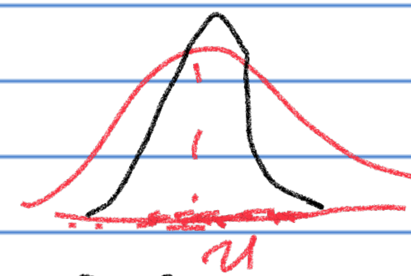
$$\chi^2_2 = \text{Gamma}\left(\frac{2}{2}, \beta=2\right) = \exp(\beta=2)$$

PDF



PDF of χ^2_n

Student's Theorem



$$X_1, \cdots, X_n \overset{IId}{\sim} N(\mu, \sigma^2)$$

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$V = \sum_{i=1}^{n} (X_i - \bar{X})^2, \quad S^2 = \frac{V}{n-1} = \frac{SS_{yy}}{n-1}$$

SS

deviation

another notation for V.

1) $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

2) $\bar{X}$ & S^2 indep.

3) $\frac{V}{\sigma^2} \sim \chi^2_{n-1}$ ← standardization.

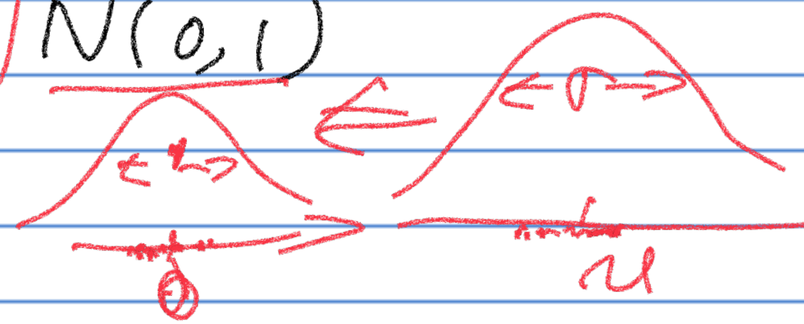
Pf:

$$\bar{Z} = \frac{\sum Z_i}{n}$$

Let $Z_i = \frac{X_i - \mu}{\sigma}$, i.e., $X_i = \mu + \sigma Z_i$

$Z_1, \dots, Z_n \overset{IID}{\sim} N(0,1)$

$\bar{X} = \mu + \sigma \cdot \bar{Z}$



$E(\bar{Z}) = 0$, $V(\bar{Z}) = \frac{1}{n} V(Z_i) = \frac{1}{n}$

$\bar{Z} \sim N(0, \frac{1}{n}) \Rightarrow \sqrt{n}\,\bar{Z} \sim N(0,1)$

$\bar{X} \sim N(\mu, \frac{1}{n}\sigma^2)$.

$$X_i = \mu + \sigma z_i$$

$$\sum X_i = \sum (\mu + \sigma z_i)$$

$$= n \cdot \mu + \sigma \cdot \sum z_i$$

$$\frac{\sum X_i}{n} = \mu + \sigma \cdot \bar{z}$$

$$\bar{X} = \mu + \sigma \bar{z}$$

$$V = \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$= \sum_{i=1}^{n} \left[\underline{\mu + \sigma Z_i} - \underline{(\mu + \sigma \bar{Z})} \right]^2$$

$$= \sigma^2 \sum_{i=1}^{n} (Z_i - \bar{Z})^2$$

$$\frac{V}{\sigma^2} = \sum_{i=1}^{n} (Z_i - \bar{Z})^2 \equiv V_Z$$

$$\bar{z} = \sum_{j=1}^{n} z_j / n, \quad E(\bar{z}) = 0, \quad E(z_i - \bar{z}) = 0$$

$$= E(z_i) - E(\bar{z}) = 0 - 0 = 0$$

$$\mathrm{Cov}(z_i - \bar{z}, \bar{z})$$

$$= E\left((z_i - \bar{z}) \cdot \bar{z}\right)$$

$$= E(z_i \bar{z}) - E(\bar{z}^2)$$

$$= \frac{1}{n} E\left(\sum_{j=1}^{n} z_i z_j\right) - V(\bar{z})$$

$$= \frac{1}{n} \sum_{j=1}^{n} E(z_i z_j) - V(\bar{z})$$

$\rightarrow 1 \quad i = j$

$\rightarrow 0 \quad \text{o.w.}$

$$= \frac{1}{n} \times 1 - \frac{1}{n} = 0$$

$$\mathrm{Cov}(X, Y)$$

$$= E(X \cdot Y)$$

$$- E(X) \cdot E(Y)$$

$$E(z_i^2) = 1$$

$$\mathrm{Cov}(\bar{z},\ z_i - \bar{z}) = 0.$$

$$\Rightarrow \bar{z} \perp z_i - \bar{z} \text{ for all } i = 1, \dots, n$$

$$\bar{z} \perp \sum_{i=1}^{n} (z_i - \bar{z})^2 = V_z$$

$$\bar{X} = \mu + \sigma \bar{z} \perp V = \sigma^2 \sum_{i=1}^{n} (z_i - \bar{z})^2$$

$\bar{z}$ and $z_i - \bar{z}$ are normal.

$$\sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n} \left(z_i - \bar{z} + \bar{z} \right)^2$$

$$\sim \chi^2_n$$

$$= \sum_{i=1}^{n} (z_i - \bar{z})^2 + n \cdot \bar{z}^2$$

$$= V_z + G$$

$\sqrt{n}\cdot\bar{z} \sim N(0,1)$

$G \sim \chi^2_1$

$$M_{V_z + G}(t) = M_{V_z}(t) \cdot M_G(t)$$

$$(1-2t)^{-\frac{n}{2}} = M_{V_z}(t) \cdot (1-2t)^{-\frac{1}{2}}$$

$$M_{V_z}(t) = (1-2t)^{-\frac{n-1}{2}}$$

So $V_z \sim \chi^2_{n-1}$