STAT 342

Mathematical Statistics

Lecture 19

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plan:

1) t & F $\Big\}$ ch 3.

2) Mixture distribution

3) Lin. dep Σ, Limiting distribution

Def of t :

Let $Z \sim N(0,1)$. $V \sim \chi^2_n$

Let $T = \dfrac{Z}{\sqrt{\frac{V}{n}}}$

We say $T \sim t_n$

We can use the ghosting method to derive the p.D.F. of t

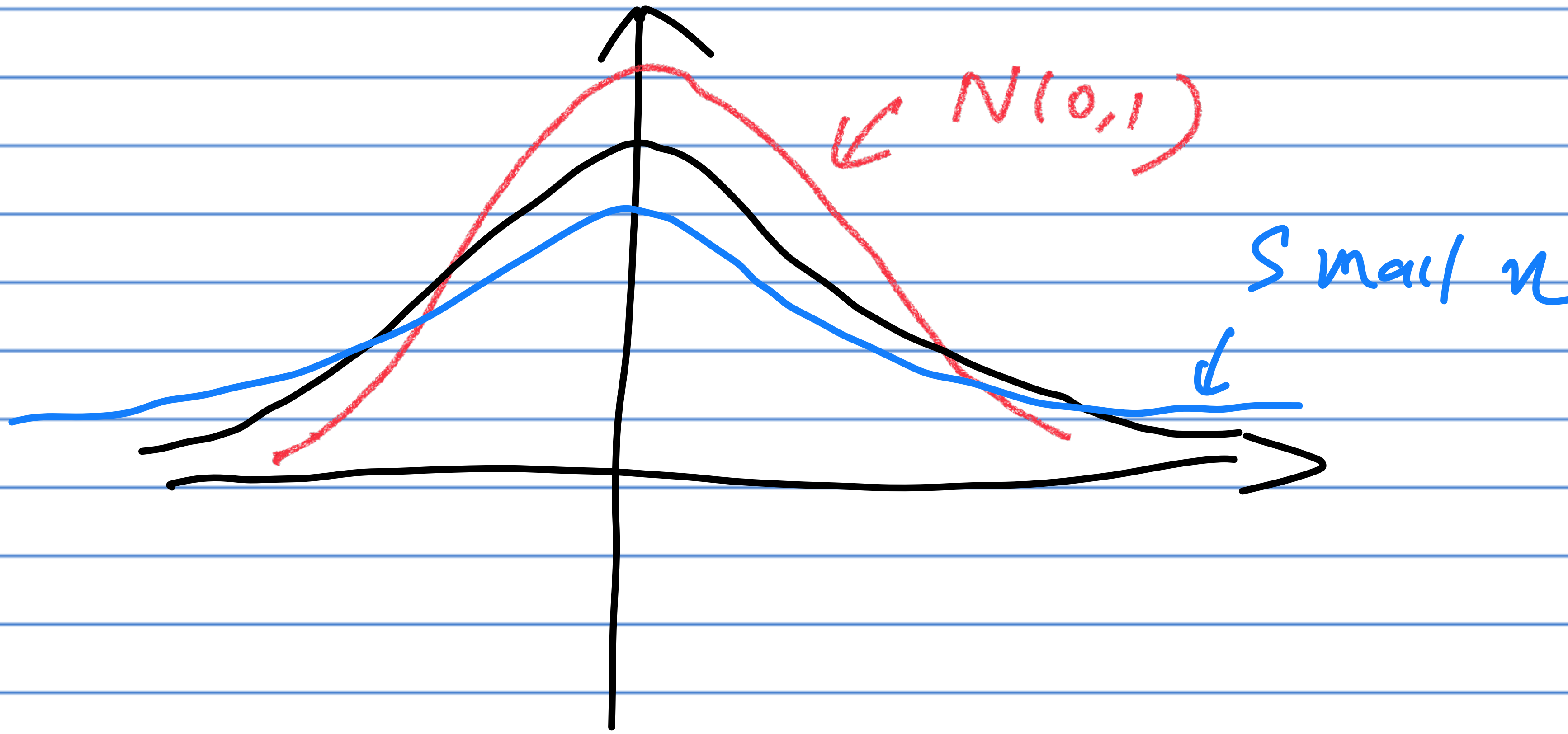
P.D.F.

$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{n}{2}\right)\sqrt{n}} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$n = 1$

$$f(t) \propto \left(1+t^2\right)^{-1}$$, Cauchy distribution

$$f(t) \propto \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} \xrightarrow{n\to\infty} e^{-\frac{t^2}{2}}$$


N(0,1)
small n

Mean & Variance

$$E(T^k) = E\left(\frac{Z^k}{\left(\frac{V}{n}\right)^{\frac{k}{2}}}\right)$$

Z & V indep

$$= n^{\frac{k}{2}} \cdot E(Z^k) \cdot \boxed{E\left(V^{-\frac{k}{2}}\right)}$$

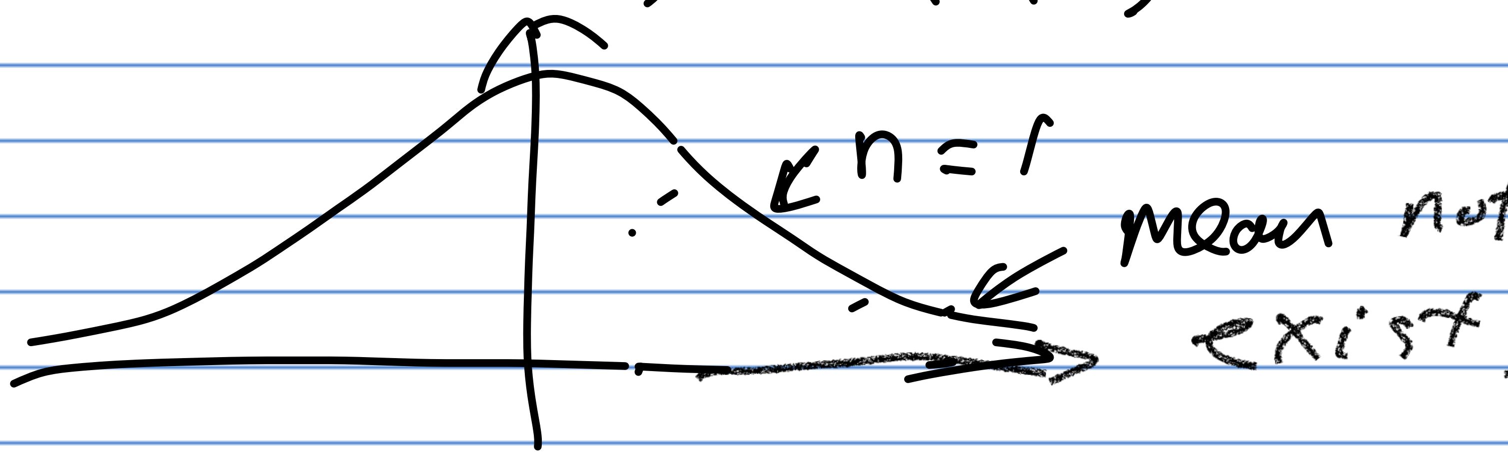
$$= n^{\frac{k}{2}} \cdot E(Z^k) \cdot \frac{\Gamma\left(\frac{n}{2} - \frac{k}{2}\right) \cdot 2^{-\frac{k}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

when $n > k$

$E(T^k) < +\infty$.

$k = 1$, when $n > 1$, $E(T) < +\infty$

when $n \le 1$, $E(|T|) = +\infty$



$$V(T) = \frac{n}{n-2}. \quad \text{when } n > 2$$

$$V(T) = +\infty \quad \text{when } n \leq 2.$$

An example:

$$X_1, \dots, X_n \overset{IID}{\sim} N(\mu, \sigma^2)$$

$$\bar{X} \sim N(\mu, \sigma^2/n) \Leftrightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

Z

$$\frac{V}{\sigma^2} \sim \chi^2_{n-1}$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$T = \frac{(\bar{X} - \mu) / \frac{\cancel{\sigma}}{\sqrt{n}}}{\sqrt{\frac{\cancel{(n-1)} S^2 / \cancel{\sigma^2}}{\cancel{n-1}}}} \sim t_{n-1}$$

$\leftarrow Z$

$\leftarrow V$

$$= \frac{\bar{X} - \mu}{\sqrt{\frac{S^2}{n}}} \sim t_{n-1}$$

F - distribution

Let $X \sim \chi^2_{n_1}$, $Y \sim \chi^2_{n_2}$

$X \perp Y$

Let $F = \dfrac{\frac{X}{n_1}}{\frac{Y}{n_2}} \sim F_{n_1, n_2}$

num. denom.

Mean

$$E(F) = E\left(\frac{\frac{X}{n_1}}{\frac{Y}{n_2}}\right)$$

$\frac{X}{n_1} \approx 1$, $\frac{Y}{n_2} \approx 1.$

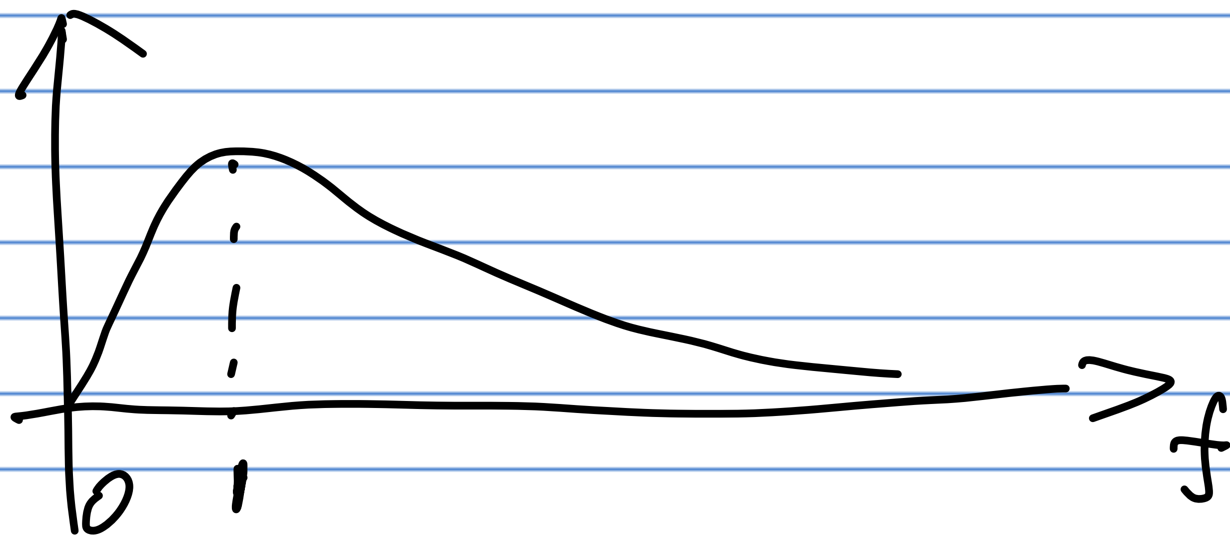
$$= \frac{n_2}{n_1} \cdot E(X) \cdot E(Y^{-1})$$

$$= \frac{n_2}{n_1} \cdot n_1 \cdot 2^{-1} \cdot \frac{\Gamma\left(\frac{n_2}{2} - 1\right)}{\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \frac{n_2}{n_2 - 2} \approx 1, \text{ when } n_2 > 2,\ n_1 > 1$$

Not Related to n_1 as long as $n_1 > 1$

Shape of P.D.F. of F



An example:

$$X_1, \dots, X_{n_1} \overset{\text{IID}}{\sim} N(\mu_1, \sigma_1^2)$$

$$Y_1, \dots, Y_{n_2} \overset{\text{IID}}{\sim} N(\mu_2, \sigma_2^2)$$

$$S_X^2 = \sum (X_i - \bar{X})^2 / (n_1 - 1)$$

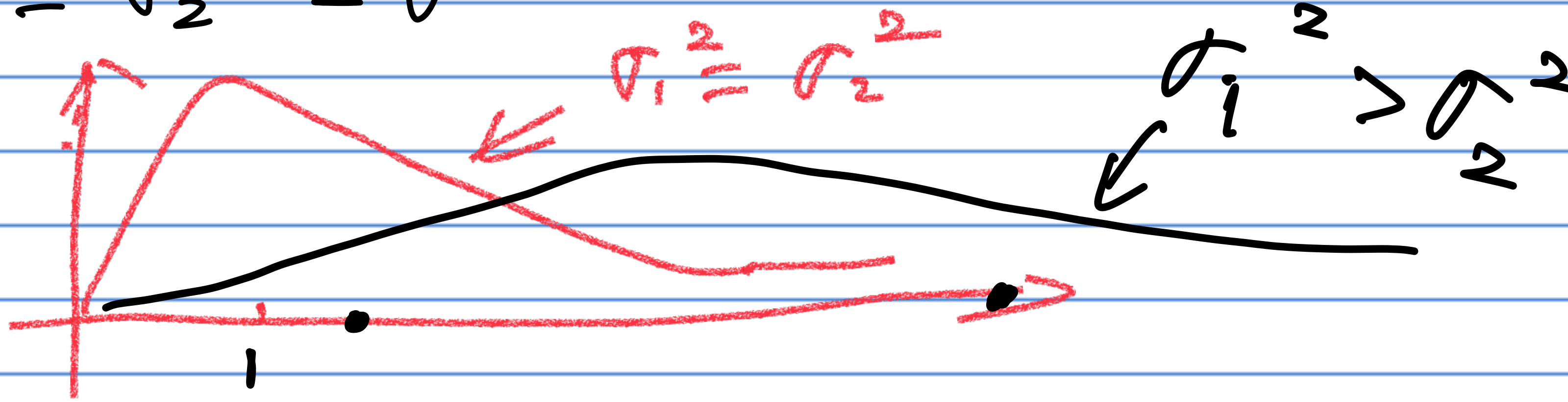
$$S_Y^2 = \sum (Y_i - \bar{Y})^2 / (n_2 - 1)$$

$$F = \frac{S_X^2}{S_Y^2} = \frac{\frac{(n_1-1)S_X^2}{\sigma^2}/(n_1-1)}{\frac{(n_2-1)S_Y^2}{\sigma^2}/(n_2-1)} \sim F_{n_1,n_2}$$

$\frac{(n_1-1)S_X^2}{\sigma^2} \sim \chi^2_{n_1-1}$

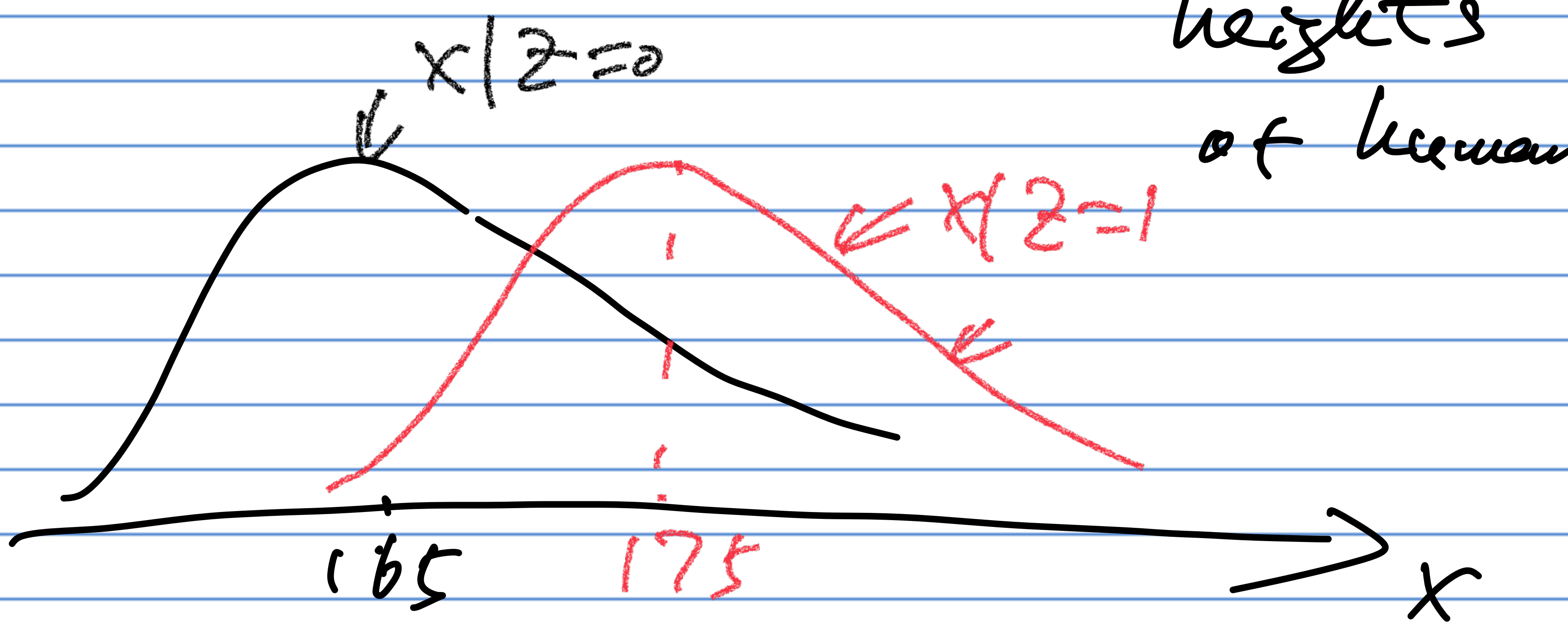
$\frac{(n_2-1)S_Y^2}{\sigma^2} \sim \chi^2_{n_2-1}$

When $\sigma_1^2 = \sigma_2^2 = \sigma^2$



Mixture distribution. (Sec 5.7)

Mixture of Normals



$$Z \sim \mathrm{Bern}(p)$$

$$X \mid Z=1 \sim N(\mu_1, \sigma_1^2)$$

$$X \mid Z=0 \sim N(\mu_0, \sigma_0^2)$$

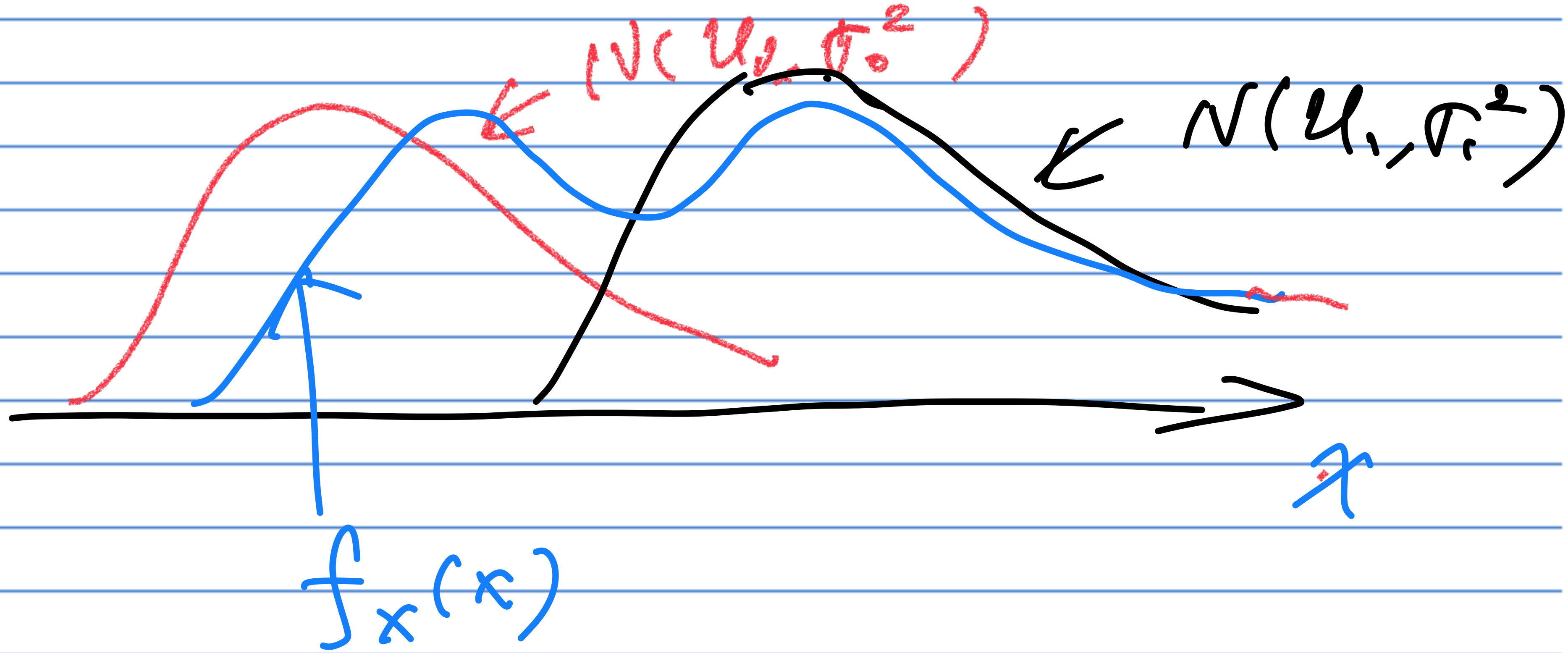
$$f(x, z) = f(x \mid z) \cdot p(z)$$

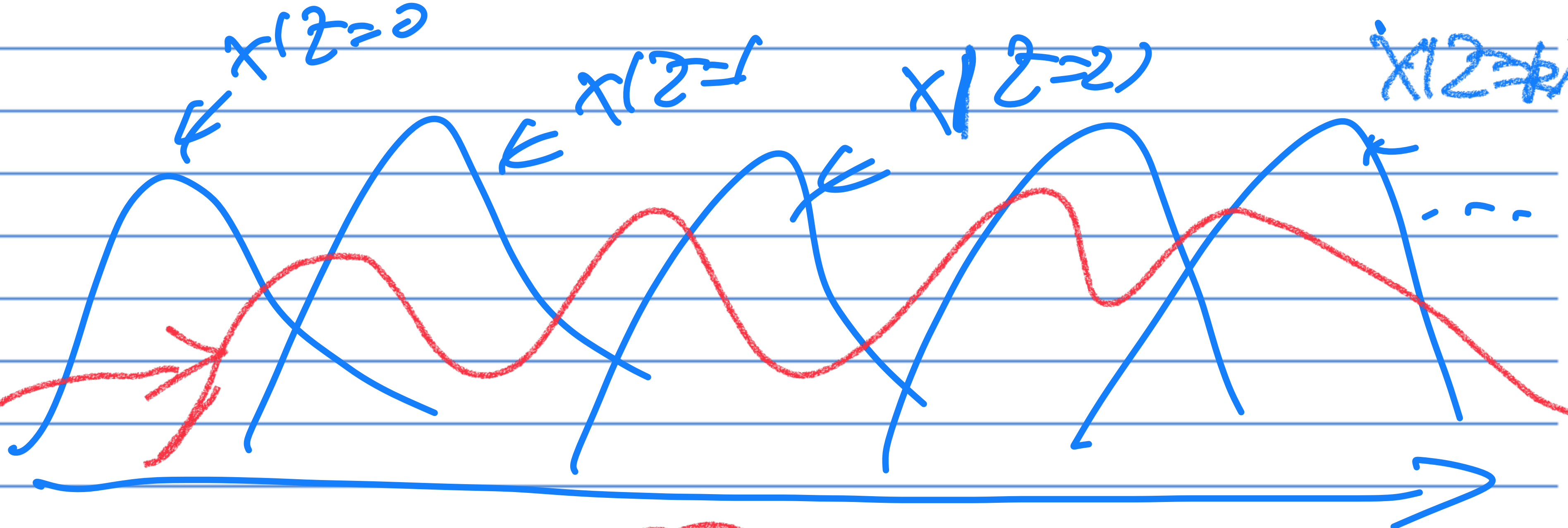
$$f(x) = \sum_{z=0}^{1} f(x \mid z) \cdot p(z)$$

$$f(x) = \underset{p(z=1)}{\underbrace{p}} \cdot \underset{f(x \mid z=1)}{\underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}}}$$

$$+ (1-p) \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{(x-\mu_0)^2}{2\sigma_0^2}}$$

Mixture of two normals.


$N(\mu_2, \sigma_2^2)$
$N(\mu_1, \sigma_1^2)$
x
$f_X(x)$



$P(z=i) = \pi_i$

$X|z=i \sim N(\mu_i, \sigma_i^2)$

X is a mixture of normals

Scale Mixture Normal (t dist.)

$$\frac{Z}{\sqrt{\frac{V}{n}}} \sim t_n \quad \text{where } Z \sim N(0,1)$$

$$V \sim \chi^2_n$$

$$X = \frac{Z}{\sqrt{\frac{V}{n}}} = Z \cdot \sigma \text{ where}$$

$$\sigma^2 = \left(\frac{V}{n}\right)^{-1} \Longleftrightarrow \sigma = \frac{1}{\sqrt{\frac{V}{n}}}$$

$X \mid \sigma^2 \sim N(0, \sigma^2)$

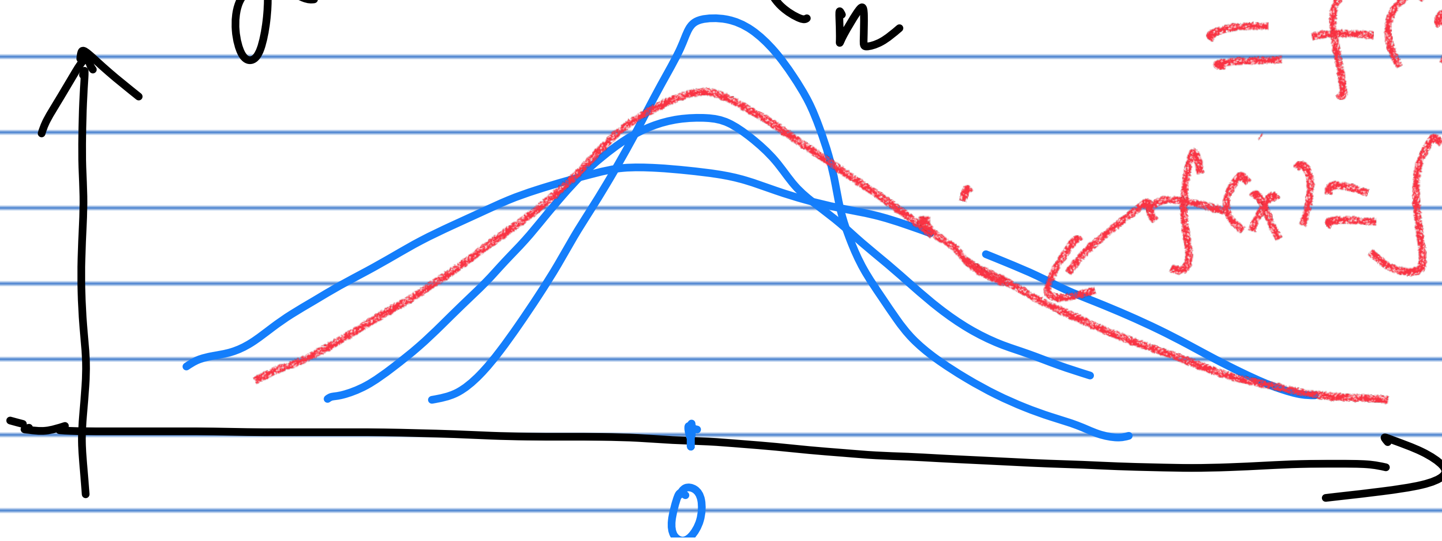
$(\sigma^2)^{-1} \sim \frac{1}{n} \chi^2_n$ [scaled-χ^2_n]

or $\frac{n}{\sigma^2} \sim \chi^2_n$

$f(x, \sigma^2)$

$= f(x \mid \sigma^2) f(\sigma^2)$

$f(x) = \int f(x, \sigma^2)\, d\sigma^2$



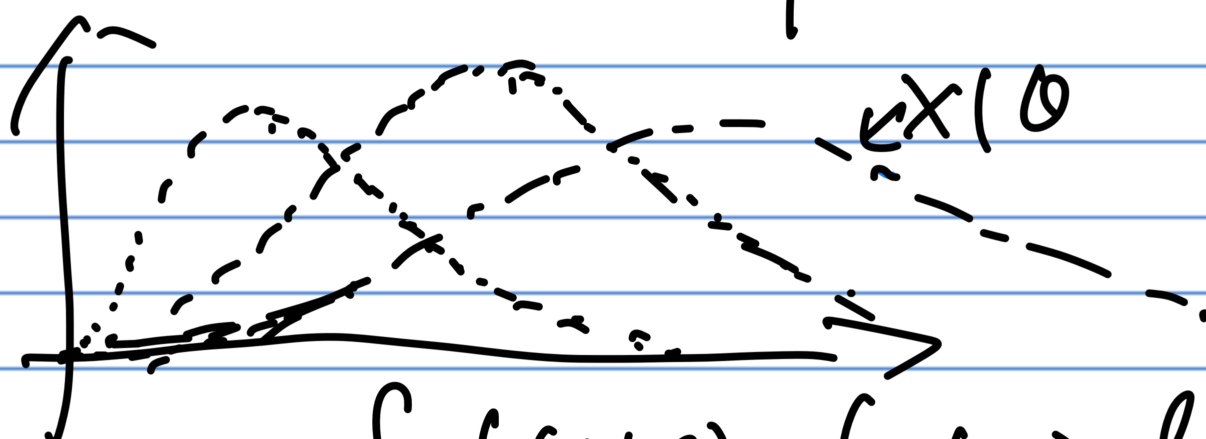
Mixture of Poissons

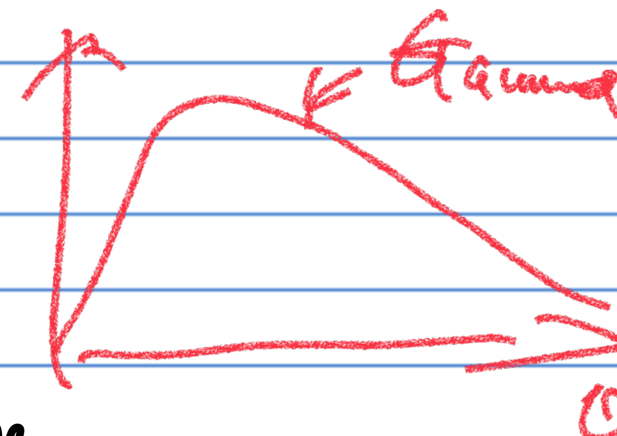
$X|\theta \sim \text{Poisson}(\underline{\theta})$ [discrete]

$\underline{\theta} \sim \underline{\text{Gamma}}\left(\alpha = r, \ \beta = \frac{1-p}{p}\right)$

Cont.

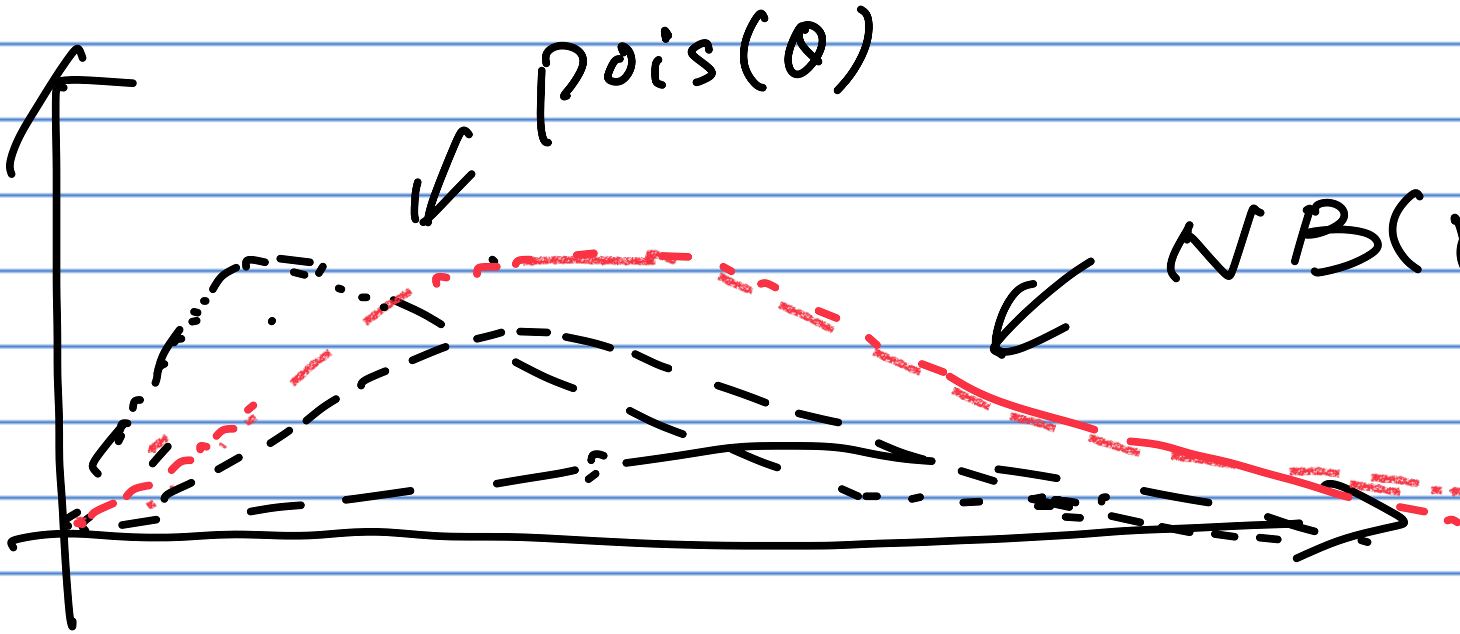
$E(\theta) = r \cdot \frac{1-p}{p}.$





$f(x) = \int f(x|\theta) \cdot f_\theta(\theta)\, d\theta$

$X \sim \underline{NB}(r, p)$



Mixture of Binomial

$X \mid N \sim \text{Binomial}(N, p)$ discrete

$N \sim \text{poisson}(\lambda)$ discrete.

$X \sim \text{poisson}(\lambda p)$