# STAT 342
# Mathematical Statistics

## Lecture 00

Longhai Li, November ??, 2010

plan: (Sec 5.1)

1) Definition of Convergence in prob.

2) Law of Large Numbers

3) Rules of Conv. in Prob

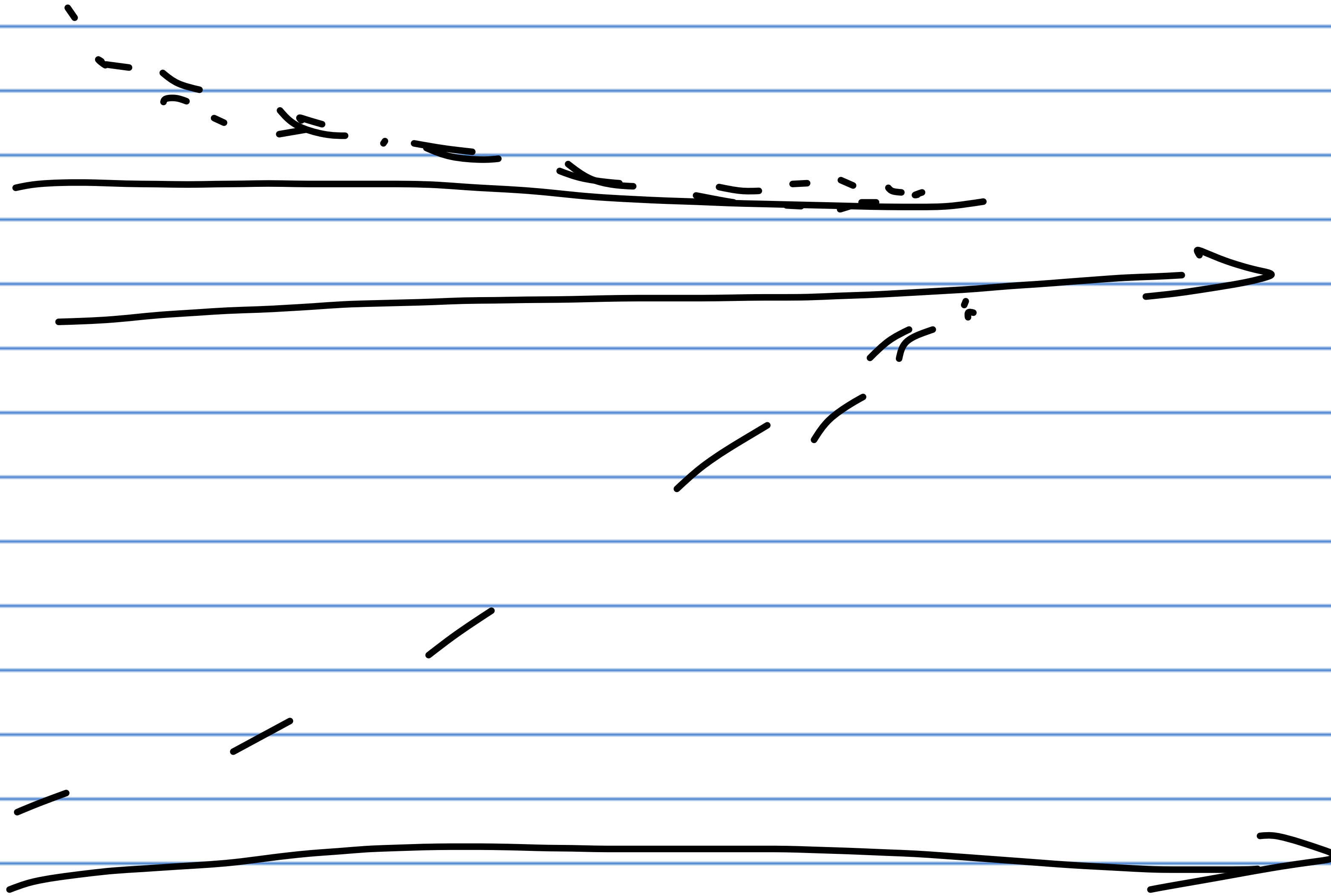
4) Consistency of $\bar{X}$ & $S^2$.

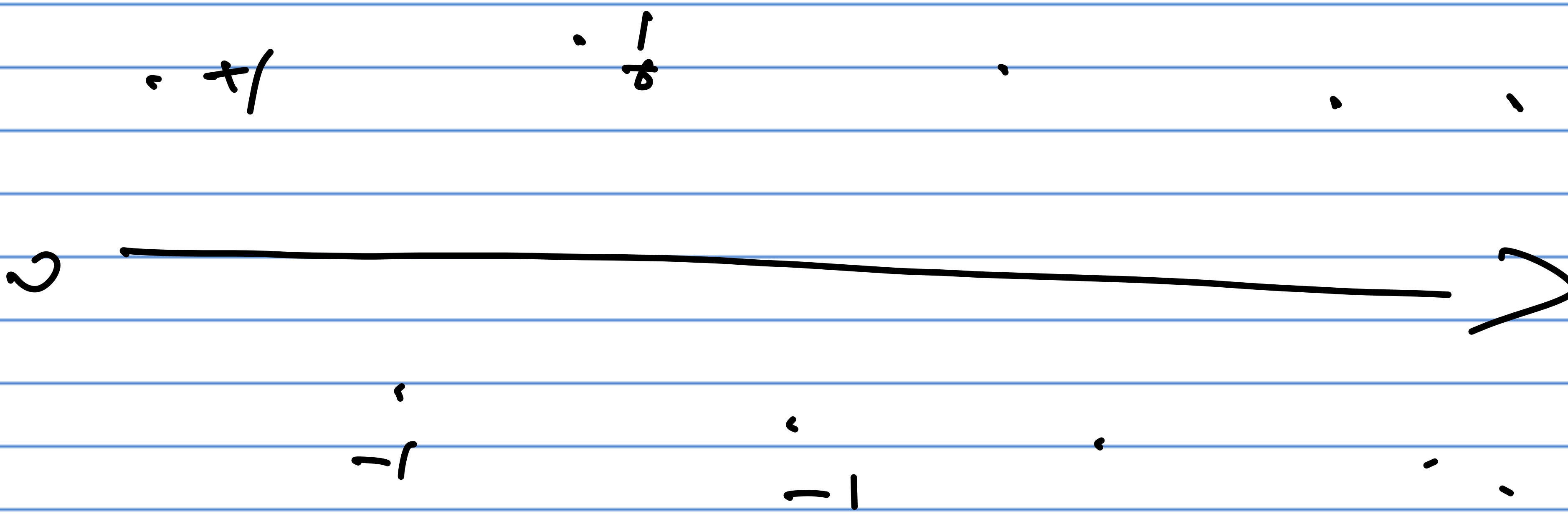

$a_n = \frac{1}{n} \longrightarrow 0$ as $n \to +\infty$

$\forall \varepsilon > 0$, $\exists$ $N$, when $n > N$, $|a_n - a| < \varepsilon$

exist

then we say $a_n \to a$

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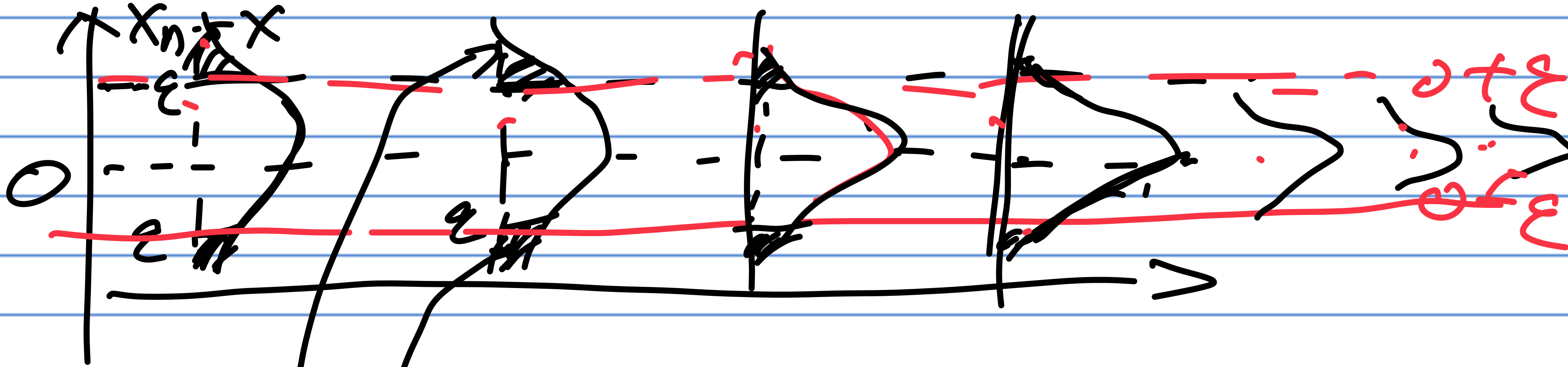
Def of $\xrightarrow{P}$ :

We say $X_n \xrightarrow{P} X$ if

$\forall \varepsilon > 0$

$$\lim_{n \to +\infty} P(|X_n - X| > \varepsilon) = 0$$

or $X_n - X \xrightarrow{P} 0$




$X_n - X \xrightarrow{P} 0$

$$P(|X_n - X| > \varepsilon) \longrightarrow 0, \ \forall \varepsilon$$

$\overset{\text{Def}}{\Longleftrightarrow} \quad X_n - X \xrightarrow{P} 0$ or $X_n \xrightarrow{P} X$

## Law of Large Number (Weak)

Thm:

suppose $X_1, X_2, \dots$ IID

and $V(X_i) = \sigma^2 < +\infty$, $E(X_i) = \mu$

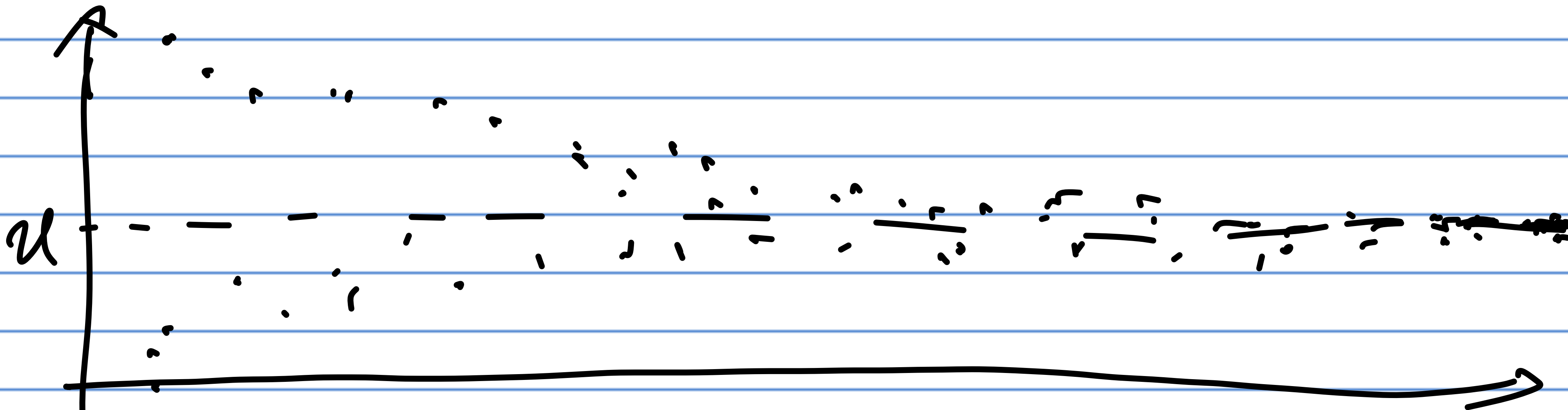
Let $$\bar{X}_n = \frac{X_1 + \dots + X_n}{n}$$

then $\bar{X}_n \xrightarrow{p} \mu$.

pf: $\forall \varepsilon > 0$

$$P(|\bar{X}_n - \mu| > \varepsilon) \le \frac{V(\bar{X}_n)}{\varepsilon^2}.$$

$$= \frac{\frac{1}{n} V(X_i)}{\varepsilon^2} = \frac{1}{n}\frac{\sigma^2}{\varepsilon^2} \longrightarrow 0$$




$$\bar{X}_n \xrightarrow{P} \mu$$

Strong Law of L.N.: $\mu < +\infty$

Consistency of estimator:

$\theta$ — unknown par.

$\hat{\theta}_n(X_1, \cdots, X_n)$ — an estimator.

e.g. $\theta = E(X_i)$, $\hat{\theta}_n = \bar{X}_n$

We say $\hat{\theta}_n$ is consistent if

$$\hat{\theta}_n \xrightarrow{P} \theta .$$

Some basic rules about conv. in prob.

Thm: $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y$

Then $X_n + Y_n \xrightarrow{P} X+Y$.

No assumption about $X_n$ & $Y_n$.

Pf: $\forall \varepsilon > 0$

$$P(|X_n + Y_n - X - Y| > \varepsilon) \to 0 \ ?$$

$\varepsilon < |a+b| \le |a| + |b|$

$$\le P(|X_n - X| + |Y_n - Y| > \varepsilon)$$

$$\le \underbrace{P\left(|X_n - X| > \frac{\varepsilon}{2}\right)}_{\downarrow\, 0} + \underbrace{P\left(|Y_n - Y| > \frac{\varepsilon}{2}\right)}_{\downarrow\, 0}$$

Thm: If $X_n \xrightarrow{P} X$, $a$ is a constant.

then $aX_n \xrightarrow{P} aX$

Thm: $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y \Rightarrow X_n \cdot Y_n \xrightarrow{P} X \cdot Y$

Conv. in prob preserves under

$+$ and $\cdot$

Thm: Continuous Mapping ✓

If $X_n \xrightarrow{p} X$, $g(\cdot)$ is a continuous function

then $g(X_n) \xrightarrow{p} g(X)$.

Pf: $X_n - X$ is small enough.

$\underline{g(X_n)} - \underline{g(X)}$ is small enough.

Example:

If $X_n \xrightarrow{P} a$

then $X_n^2 \xrightarrow{P} a^2$

$X_n + 1 \xrightarrow{P} a + 1$

$\log(X_n) \xrightarrow{P} \log(a)$

$\sqrt{X_n} \xrightarrow{P} \sqrt{a}$.

Example :

$X_1, X_2, \ldots$ are IID with

$V(X_i) < +\infty$

$$S_n^2 = \frac{\sum_{i=1}^{+\infty} (X_i - \bar{X})^2}{n-1} \xrightarrow{P} \sigma^2 \ ?$$

why ?

$$S_n^2 = \frac{1}{n-1} \cdot \left[ \sum_{i=1}^{n} (X_i - \bar{X})^2 \right]$$

$$= \frac{1}{n-1} \left[ \sum_{i=1}^{n} X_i^2 - n\bar{X}^2 \right]$$

$$= \frac{\sum_{i=1}^{n} X_i^2}{n} \cdot \frac{n}{n-1} - \frac{n}{n-1} \bar{X}^2$$

$$\xrightarrow{p} \quad E(X_i^2) \cdot 1 + 1 \, [E(X_i)]^2$$

$$= E(X_i^2) - [E(X_1)]^2 = \sigma^2$$

Exmp0

$$\frac{\sum (X_i - \bar{X})^2}{n} \xrightarrow{P} \sigma^2$$

why

$$\frac{\sum (X_i - \bar{X})^2}{n} = \frac{\sum (X_i - \bar{X})^2}{n-1} \cdot \frac{n-1}{n}$$

$$\downarrow P \qquad \downarrow$$

$$\sigma^2 \cdot 1$$

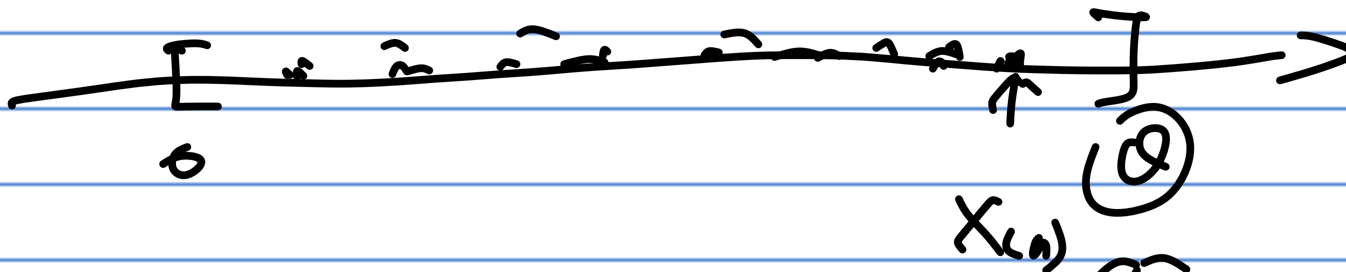
$$= \sigma^2$$

$$\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n+a} \xrightarrow{P} \sigma^2$$

$a$ is a constant unrelated to $n$.

Example:

$$X_1, \dots, X_n \overset{\text{IID}}{\sim} \text{unif}((0,\theta))$$

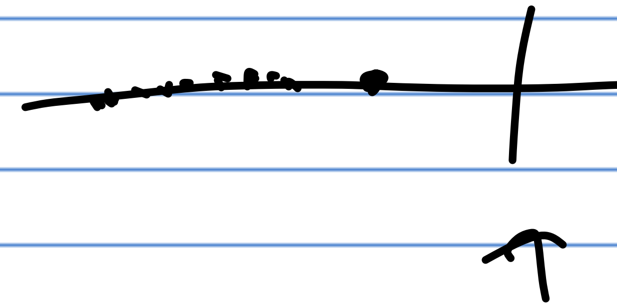



unknown.

$$f_{X_i}(x) = \frac{1}{\theta}, \text{ for } 0 < x < \theta$$

$$X_{(n)} = \max(X_1, \dots, X_n) \quad \text{order statistics}$$

$X_{(n)} \xrightarrow{P} \theta$ ?

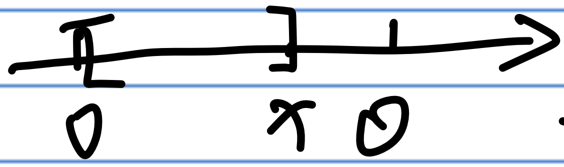



Find C.D.F. of $X_{(n)}$

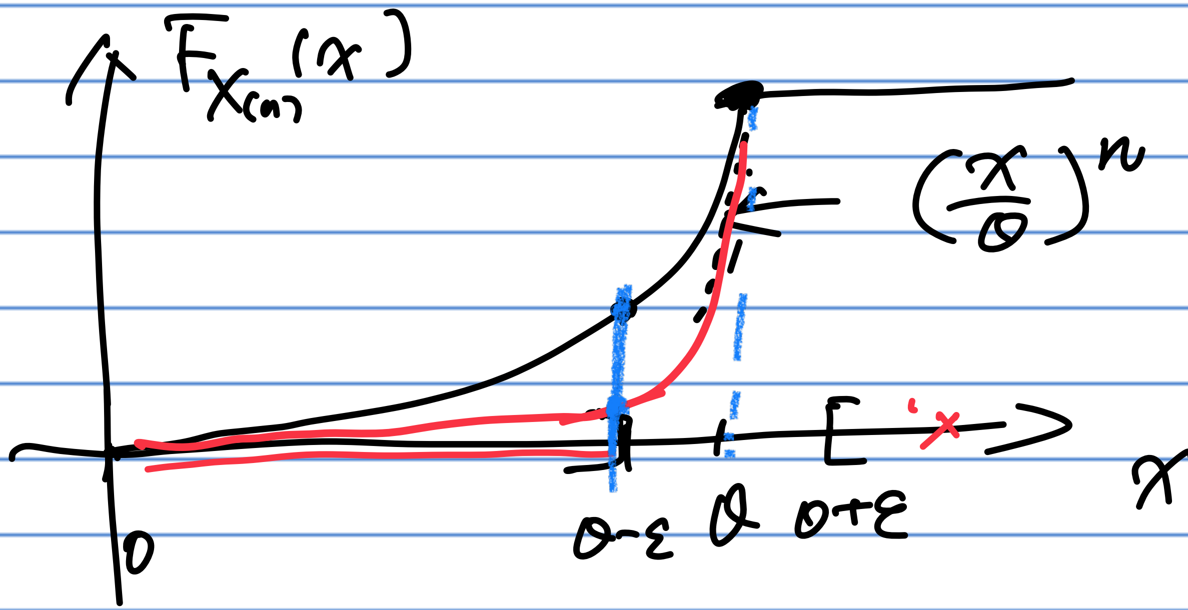
$$F_{X_{(n)}}(x) = P(X_{(n)} \le x)$$

$$= P(X_1 \le x, \ldots, X_n \le x)$$

$$= [P(X_i \le x)]^n$$




$$= \left(\frac{x}{\theta}\right)^n, \text{ for } x \in (0, \theta)$$



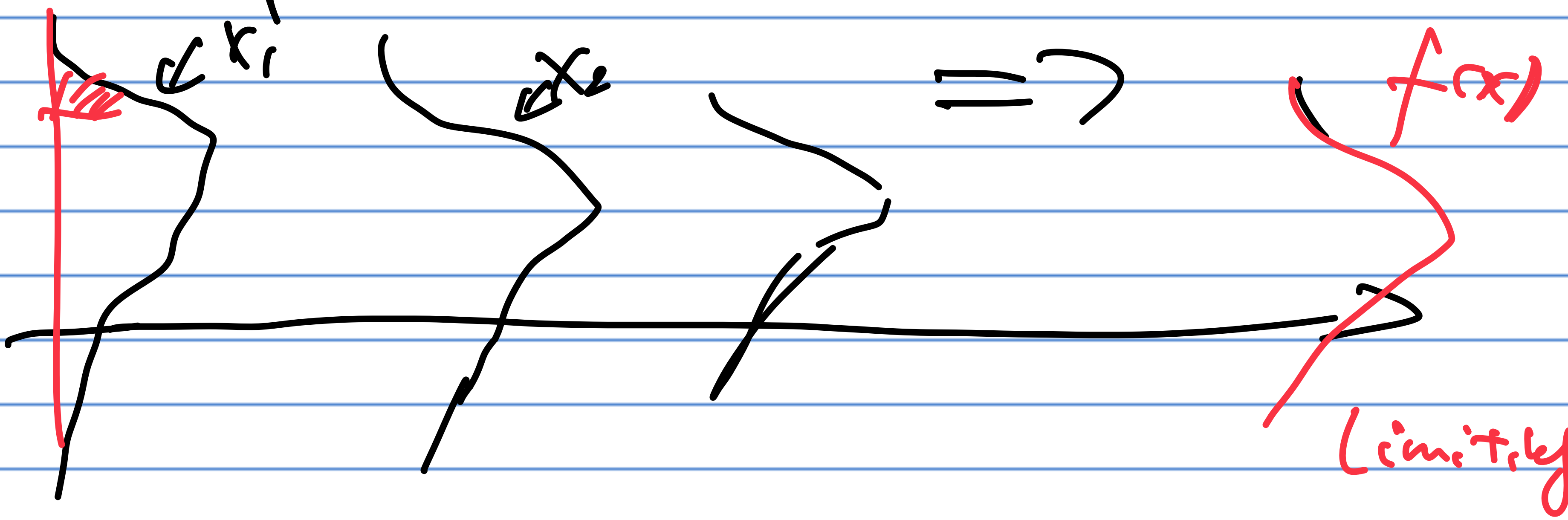

$$P(|X_{(n)}-\theta|>\varepsilon) = P(X_{(n)}<\theta-\varepsilon)$$

$$= F_{X_{(n)}}(\theta-\varepsilon) = \Big[\underbrace{\frac{\theta-\varepsilon}{\theta}}_{<1}\Big]^n \xrightarrow{n\to+\infty} 0$$

$x$

So, $X_{(n)} \xrightarrow{P} \theta$.

Convergence in distribution.




the dist. of $X_n$ ⟶ the dist of $X$.

Def:

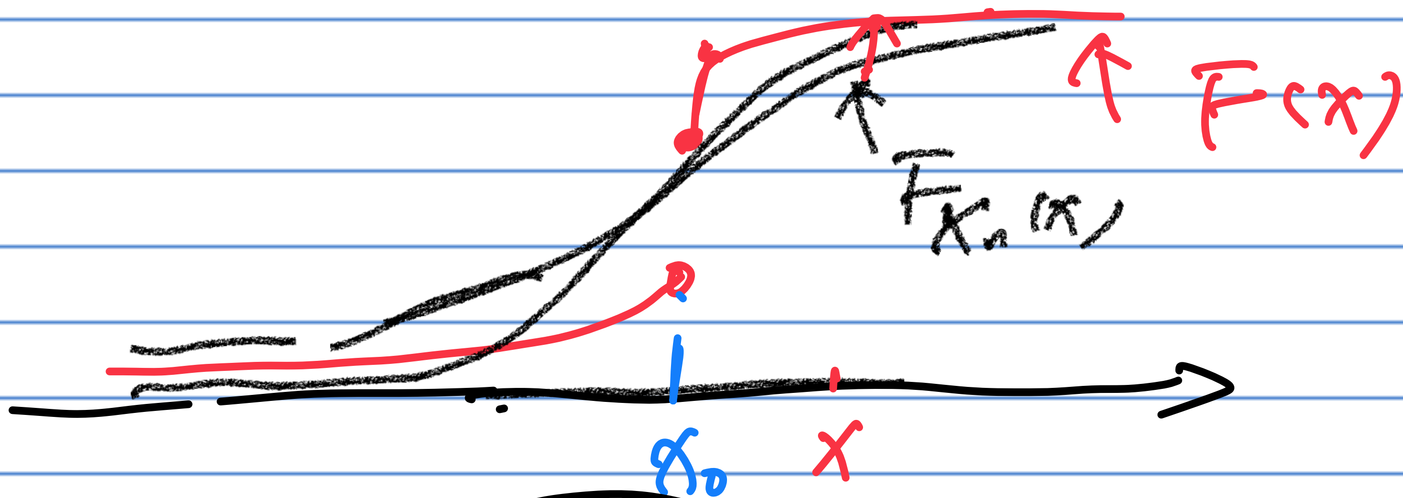
We say $X_n \xrightarrow{d} X$ or $F_{X_n} \xrightarrow{d} F_X$

if

$$F_{X_n}(x) \xrightarrow{d} F_X(x)$$

for each $x \in C(F_X)$, where

$C(F_X)$ is the collection of all continuous point of $F_X$




$$F_{X_n}(x) \xrightarrow{n\to\infty} F(x)$$

if $F$ is cont. at $x$.

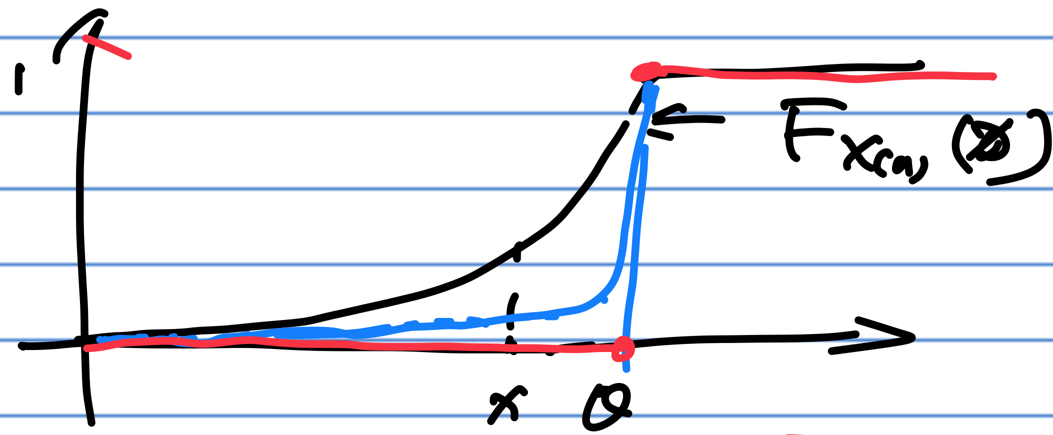
we don't care $x_0$.

Example:

$$X_{(n)} = \max(X_1, \cdots, X_n)$$

$$X_1, X_2, \cdots \overset{IID}{\sim} \text{Unif}((0, \theta))$$

$$F_{X_{(n)}}(x) = \begin{cases} \left(\frac{x}{\theta}\right)^n, & \text{if } 0 \leq x \leq \theta \\ 1, & \text{if } x \geq \theta \\ 0, & x \leq 0 \end{cases}$$




$x < \theta$

$\left(\frac{x}{\theta}\right)^n \longrightarrow 0$

$F(x) = I(x \geq \theta)$

$x = \theta$ $\left(\frac{x}{\theta}\right)^n = 1$

$F_{X_{(n)}}(x) \longrightarrow F(x)$ for all $x$

$F(x)$ is the CDF of $X = \theta$

$X_{(n)} \xrightarrow{d} \theta$. or $F_{X_{(n)}} \xrightarrow{d} F_x$