STAT 342

Mathematical Statistics

Lecture 21

Longhai Li, November 30, 2010

plan: sec 5.2 & 5.3

1. Def of Conv. in dist

2. Central Limit Theorem

3. Rules of Conv. in distribution

   - continuous mapping
   - slutsky Theorem

4. Limiting distributions about $\hat{p}$ & $\bar{X}$.

Example of Conv. in dist.

$X_1, X_2, \cdots \overset{IID}{\sim} N(\mu, \sigma^2)$

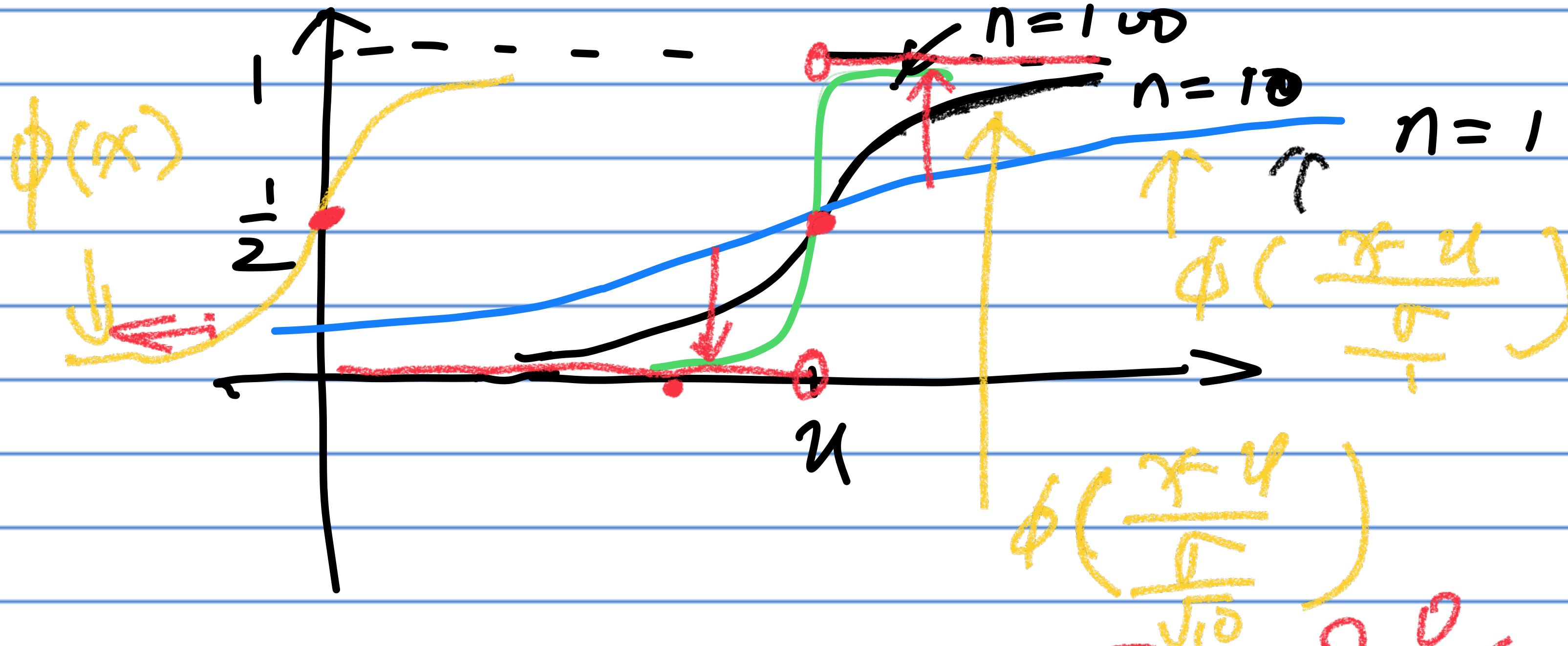
we knows that

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$F_{\bar{X}_n}(x) = P(\bar{X}_n \le x)$$

$$= P\left(Z \le \frac{x-\mu}{\frac{\sigma}{\sqrt{n}}}\right) = \phi\left(\frac{x-\mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

CDF of $N(0,1)$



$$F_{\bar{X}_n}(u) = \frac{1}{2}$$

$$F_{\bar{X}_n}(x) \to \begin{cases} 0, & x < u \\ 1, & x > u \end{cases}$$

$$\lim_{n\to\infty} F_{\bar{X}_n}(x) = F^*(x)$$

$$F^*(x) = \begin{cases} 0, & x < u \\ \frac{1}{2}, & x = u \\ 1, & x > u \end{cases}$$

NOT a C.D.F.

Let $F(x) = \begin{cases} 1, & x \geq \mu \\ 0, & x < \mu \end{cases}$

The C.D.F. of $X = \mu$.

We see that $F_{\bar{X}_n}(x) \longrightarrow F(x)$

limiting dist of $\bar{X}_n$

for all x except $x = \mu$.

We still say that $\bar{X}_n \xrightarrow{d} \mu$

In this example a continuous dist

$\rightarrow$ a discrete distribution.

A remark:

Generally, we don't define conv. in dist. in terms of conv. of P.M.F. or P.D.F. because

$$\text{cont.} \xrightarrow{d} \text{discrete}$$

$$\text{discrete} \xrightarrow{d} \text{cont.}$$

But sometimes, cont $\to$ cont

discrete $\to$ discrete.

Example of using P.D.F. to find limiting distribution.

$T_n \sim t_n$, n is the degree freedom

$$f_{T_n}(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)\sqrt{n}}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$

$\frac{1}{\sqrt{\pi}}$ $\qquad$ Stirling $\frac{1}{\sqrt{2}}$ $\qquad$ $e^{-\frac{t^2}{2}}$

N(0,1)

This is the p.d.f. of N(0,1)

That is $T_n \to N(0,1)$ as $n \to +\infty$

$$\left(1+\frac{1}{n}\right)^{n} \longrightarrow e$$

$$\left(1+\frac{t^2}{n}\right)^{-n/2} \longrightarrow e^{-t^2/2}$$

$$\left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}}$$

$$\downarrow$$

$$1$$

Conv. in dist. using M.G.F.

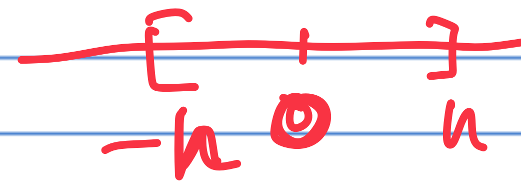
Thm:

$$X_n \xrightarrow{d} X \iff \underline{M_{X_n}(t)} \to \underline{M_X(t)}$$

for $|t| < h$, for some h.

$F_{X_n}(x) \to F_X(x)$

for $x \in C(F_X)$



STAT 841

Example:

$$X_n \sim \text{Binomial}\left(n,\ p_n = \frac{\lambda}{n}\right) \quad \text{i.e. } np_n \to \lambda$$

$$M_{X_n}(t) = \left(1 - p_n + p_n e^t\right)^n$$

$$= \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^t\right)^n$$

$$= \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n \qquad \left[\left(1 + \frac{a}{n}\right)^{\frac{n}{a}}\right]^a \to e^a$$

$$\longrightarrow e^{\lambda(e^t-1)} =: M_X(t)$$

where $X \sim \text{poisson}(\lambda)$

Central Limit Theorem

$X_1, X_2, \cdots$ IID with $\mu = E(X_i) < +\infty$, $\sigma^2 = V(X_i) < +\infty$

Let $\bar{X}_n = \dfrac{X_1 + \cdots + X_n}{n}$. Note that $\begin{cases} E(\bar{X}_n) = \mu \\ V(\bar{X}_n) = \dfrac{\sigma^2}{n} \end{cases}$

Then $\dfrac{(\bar{X}_n - \mu)}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} \underline{N(0,1)}$

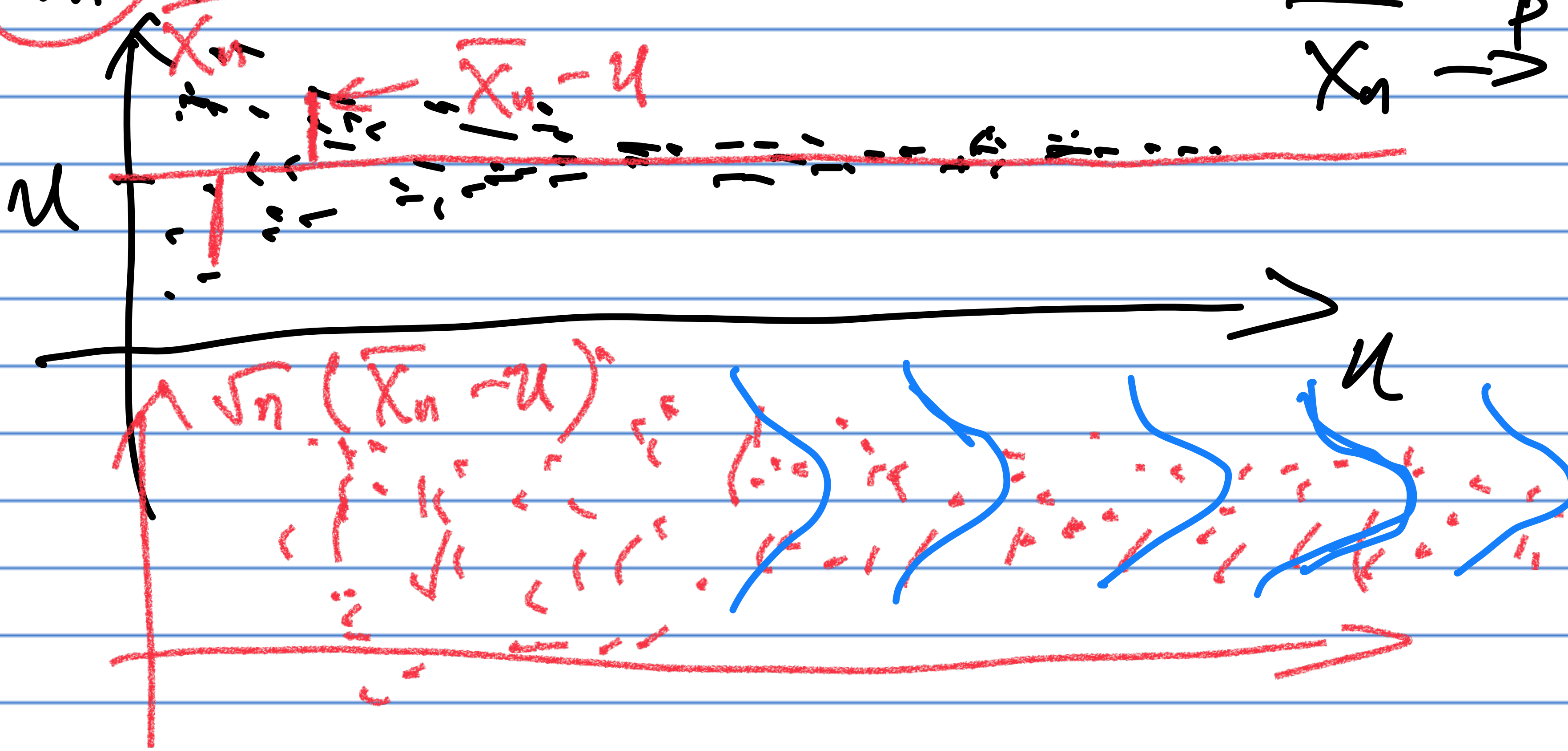
or $\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{d} N(0,1)$

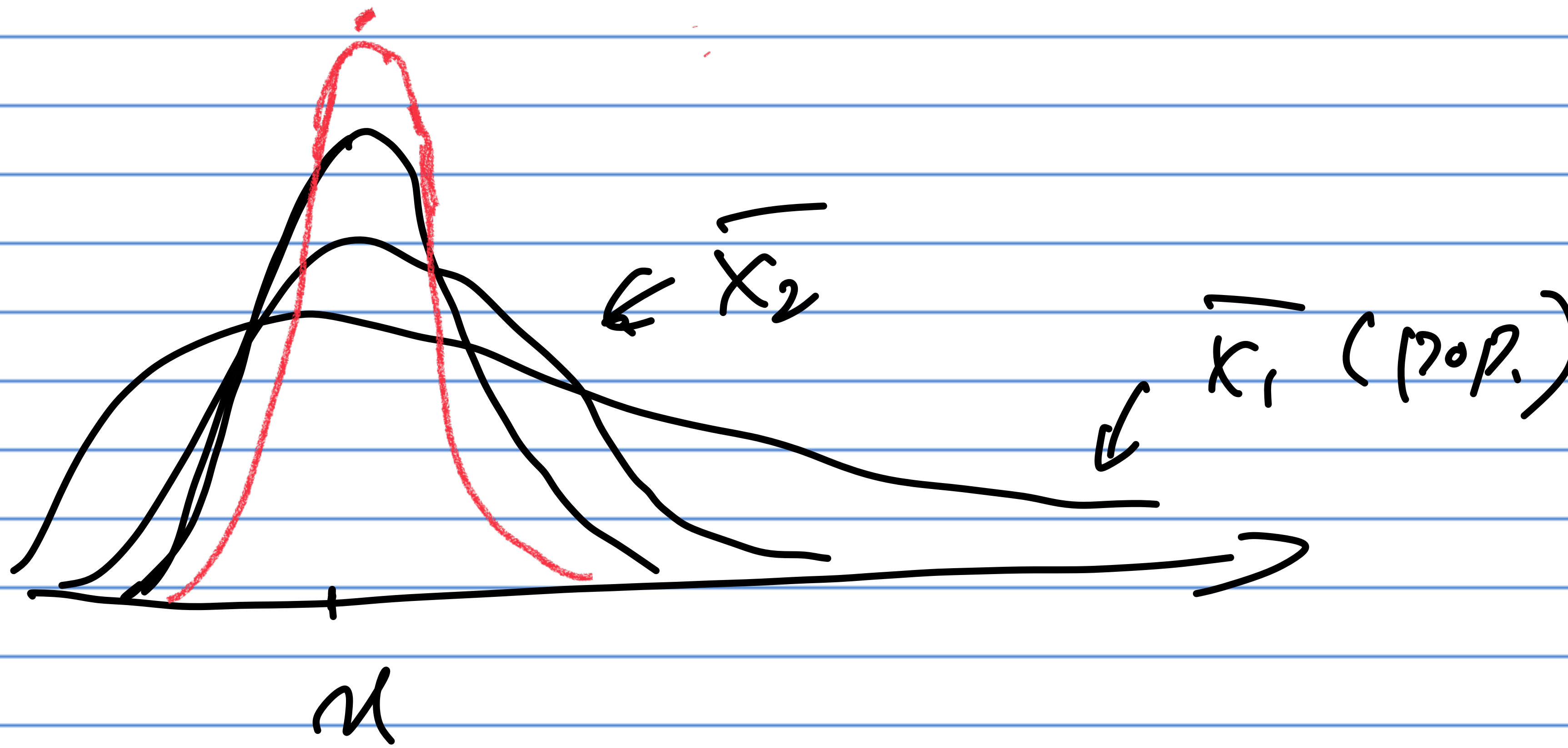
or $\bar{X}_n \overset{\text{approx}}{\sim} N\left(\mu, \dfrac{\sigma^2}{n}\right)$

$\sigma = 1$

$$\sqrt{n}(\bar{X}_n - \mu) \to N(0,1)$$

$$\bar{X}_n \xrightarrow{p} \mu$$




$\bar{X}_2$
$\bar{X}_1$ (pop.)
μ

Pf:

Let $z_i = \dfrac{X_i - \mu}{\sigma}$, then $E(z_i) = 0$, $V(z_i) = 1$

$\bar{Z}_n = (z_1 + \cdots + z_n)/n$

$$\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{(\sigma \bar{Z}_n + \mu) - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \sqrt{n}\, \bar{Z}_n$$

$$M_{\sqrt{n}\bar{Z}}(t) = E\left(e^{\sqrt{n}t\bar{Z}}\right) \qquad \bar{Z} = \frac{Z_1 + \cdots + Z_n}{n}$$

$$= E\left(e^{\frac{t}{\sqrt{n}}(Z_1 + \cdots + Z_n)}\right)$$

$$= \left(M_{Z_i}\left(\frac{t}{\sqrt{n}}\right)\right)^n \qquad \frac{t}{\sqrt{n}} \to 0$$

$$\left(1 + \frac{t^2}{2n}\right) \qquad = \left(1 + \cancel{\frac{t}{\sqrt{n}}\underbrace{E(Z_i)}_{=0}} + \frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2 \underbrace{\cancel{E(Z_i^2)}}_{=1} + \cancel{o\left(\frac{t^2}{n}\right)}\right)^n$$

$$M'_{Z_i}(0) = E(Z_i) \qquad M''_{Z_i}(0) = E(Z_i^2)$$

$$\approx \left(1 + \frac{t^2}{2n}\right)^n \to e^{\frac{t^2}{2}}$$

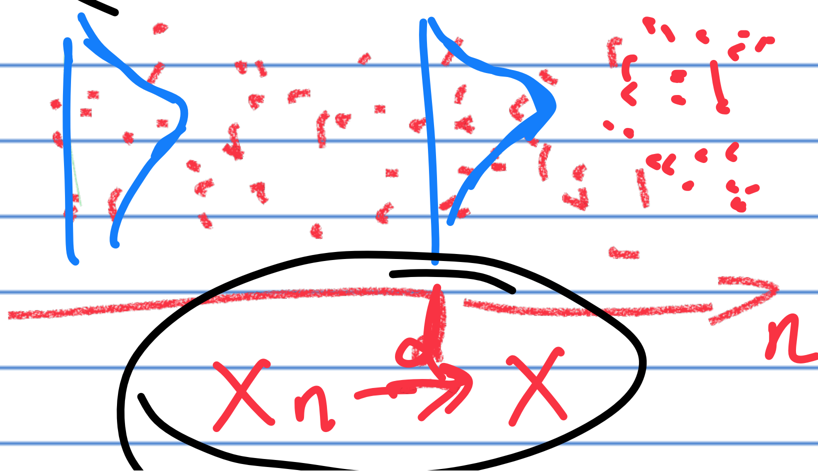
This is the MGF of $N(0,1)$.

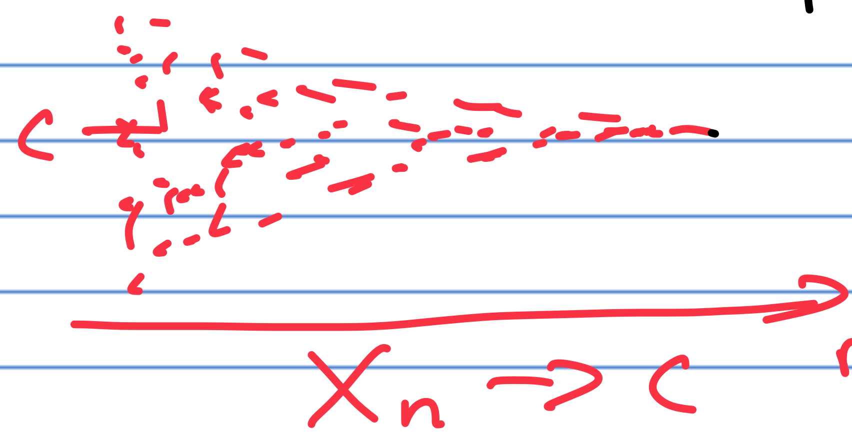
Some rules of Conv. in distributions

Thm: 1) $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$ ✓

2) $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$

Convergence to a point in prob $\Leftrightarrow$ - - - - - - - in distribution

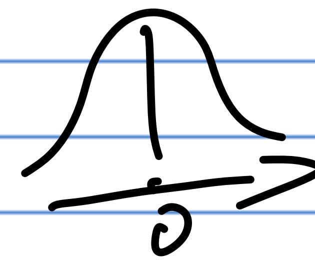




Example:

why $X_n \xrightarrow{d} X \not\Rightarrow X_n \xrightarrow{p} X$?

$X \sim N(0,1)$, $X \overset{d}{=} -X$



$$X_n = \begin{cases} X, & n \text{ is odd} \\ -X, & n \text{ is even} \end{cases}$$

$X_n \overset{d}{=} X \ (N(0,1))$

$X_n \xrightarrow{d} X$. $X_n - X = -2X$ when n is even

Thm: Continuous Mapping

Supp. $X_n \xrightarrow{d} X$, g is a continuous fn.

Then $g(X_n) \xrightarrow{d} g(X)$

✓ Thm:

Suppose $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{d} c$ (a constant)

$g(x, y)$ is a continuous function

Then $g(X_n, Y_n) \xrightarrow{d} g(X, c)$ ✓

Slutky's Theorem!

why $g(X_n, Y_n) \xrightarrow{d} g(X, Y)$ doesn't hold in general for all cont. function g?

Let $g(x,y) = x+y$, $X \overset{d}{=} -X$, eg. $X \sim N(0,1)$

$X_n = X$, $Y_n = -X \xrightarrow{d} Y = X$

$g(X_n, Y_n) = 0$

$g(X, Y) = \boxed{2X}$.

$g(X_n, Y_n) \not\to g(X, Y)$

Example:

Large sample dist of $\hat{p}$

Let $X_1, X_2, \cdots$ IID Bern(p)

$$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n} = \hat{p}_n \xrightarrow{P} p$$

We want to know the dist. of $\hat{p}_n - p$.

By C.L.T.,

$$\frac{\hat{p}_n - p}{\sqrt{\dfrac{p(1-p)}{n}}} \xrightarrow{d} N(0,1)$$

$$V(X_i) = p(1-p)$$

How about

$$\frac{\hat{p} - p}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}} \sim ?$$

$$\frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} = \frac{\hat{p}-p}{\sqrt{\frac{\cancel{p(1-p)}}{n}}} \cdot \frac{\sqrt{\cancel{p}(1-p)}}{\sqrt{\hat{p}(1-\hat{p})}}$$

$$\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} N(0,1)$$

$$\frac{\sqrt{p(1-p)}}{\sqrt{\hat{p}(1-\hat{p})}} \xrightarrow{P} 1 = \sqrt{\frac{p(1-p)}{p(1-p)}}$$

$$\xrightarrow{d} \underline{N(0,1)} \cdot 1$$

Example:

Suppose $X_1, X_2, \cdots$ IID with $E(X_i) = \mu$

and $V(X_i) < \infty$.

What's the limiting distribution of

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \to N(0,1)?$$

n is large.

By C.L.T., we know that

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

$$S^2 \xrightarrow{p} \sigma^2$$

$$\Rightarrow S \xrightarrow{p} \sigma$$

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \underbrace{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}_{\downarrow d \; N(0,1)} \cdot \underbrace{\frac{\sigma}{S}}_{\downarrow p \; 1}$$

$$\xrightarrow{d} N(0,1)\cdot 1 = N(0,1)$$